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CHARACTERIZING $C(X)$ AMONG INTERMEDIATE C-RINGS ON X

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Abstract. Let X be a completely regular topological space. An intermediate ring is a ring $A(X)$ of continuous functions satisfying $C^*(X) \subseteq A(X) \subseteq C(X)$. We give a characterization of $C(X)$ in terms of extensions of functions in $A(X)$ to real-compactifications of X. We also give equivalences of properties involving the closure in the real-compactifications of X of zero-sets in X; we use these equivalences to answer an open question about the correspondences of ideals in intermediate rings and z-filters on X.

1. Introduction

The ring $C(X)$ of all continuous real-valued functions on the completely regular Hausdorff space X has been characterized among its subrings in various ways. Perhaps the best known characterization is the Stone-Weierstrass Theorem: If X is compact, then any subring of $C(X)$ which contains the constants, separates points, and is uniformly closed is $C(X)$ itself [16, p. 291]. Other characterizations can be found in the book Characterizations of $C(X)$ among its Subalgebras by R. B. Burkel [2]. In this article we consider intermediate subrings of $C(X)$, that is, subrings $A(X)$ satisfying

$$C^*(X) \subseteq A(X) \subseteq C(X)$$

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where $C^*(X)$ is the subring of $C(X)$ consisting of the bounded functions. In [4], to each $A(X)$ there is a realcompact space $\upsilon_A X$ to which each $f \in A(X)$ can be continuously extended to a function f^{υ_A} defined on all of $\upsilon_A X$, and f^υ can be further extended to all of βX via the Stone extension of f, denoted f^*. A C-*ring* is a ring $A(X)$ that is isomorphic to $C(Y)$ for some completely regular space Y. In this article we give a characterization of $C(X)$ among intermediate C-rings $A(X)$ in terms of the extension of functions to $\upsilon_A X$ (Theorem 3.7), and we use this characterization to give a more direct proof of a result by Swardson [14, Corollary 4.3]. We also characterize intermediate C-rings that have the property that the closure of a zero-set in $\upsilon_A(X)$ is a zero-set (Theorem 4.2). We furthermore characterize $C^*(X)$ among intermediate rings $A(X)$ in terms of the relationship between the extension f^{υ_A} and the Stone extension f^* of f for each f (Theorem 3.10). In the setting of this paper, we give a more direct proof of a result by Rudd (the equivalence of the first two parts of [11, Theorem 1.2]) concerning the relationship between f and f^* (Theorem 3.11 in this paper). Furthermore, we give a class of intermediate rings for which the natural correspondence between ideals and z-filters maps maximal ideals to z-ultrafilters (Theorem 5.2). This answers [13, Question 4] in the negative: it is not the case that $A(X) = C(X)$ if and only if every maximal ideal in $A(X)$ is mapped by this correspondence to a z-ultrafilter on X.

2. Preliminaries

We follow the definitions and notation of [12, 13]. Let X be a completely regular space. A zero-set in X is a set of the form $\boldsymbol{Z}(f) = \{x \in X : f(x) = 0\}$ for some f in $A(X)$; $\boldsymbol{Z}[X]$ denotes the family of all zero-sets of X. It is well known (see [7]) that $\boldsymbol{Z}$ acts as a correspondence between proper ideals I in $C(X)$ and z-filters $\boldsymbol{Z}[I] = \{\boldsymbol{Z}(f) : f \in I\}$ on X. There is a different but well known mapping $\boldsymbol{E}$ mapping ideals in $C^*(X)$ to z-filters on X, given by $\boldsymbol{E}[I] = \bigcup\{\boldsymbol{E}_\epsilon(f) : f \in I, \epsilon > 0\}$, where $\boldsymbol{E}_\epsilon(f) = \{x : |f(x)| \le \epsilon\}$ (see [7]). However, $\boldsymbol{E}$ and $\boldsymbol{Z}$ fail to be correspondences for intermediate rings $A(X)$ strictly between $C^*(X)$ and $C(X)$ and z-filters on X [13, Propositions 3.2 and 4.2]. Thus extensions $\mathcal{Z}_A$ of $\boldsymbol{E}$ and $\mathfrak{Z}_A$ of $\boldsymbol{Z}$ to intermediate rings were defined as follows. If $f \in C(X)$ and $E \subseteq X$, we say that f is E-*regular* in $A(X)$ if there exists $g \in A(X)$ such that $f(x) \cdot g(x) = 1$ for all $x \in E$. We may simply write that f is E-regular if $A(X)$ is understood from context. For $f \in A(X)$ we have

$$\begin{aligned} \mathcal{Z}_A(f) &= \{E \in \boldsymbol{Z}[X] : f \text{ is } E^c\text{-regular}\} \\ \mathfrak{Z}_A(f) &= \{E \in \boldsymbol{Z}[X] : f \text{ is } H\text{-regular for all zero-sets } H \subseteq E^c\}. \end{aligned}$$

Note the difference between $\mathcal{Z}_A(f)$ and $\mathfrak{Z}_A(f)$. For example, if $A(X) = C(X)$ then $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$, the z-filter consisting of all zero-sets containing $\boldsymbol{Z}(f)$, whereas $\mathcal{Z}_A(f)$ consists of all zero-set neighborhoods of $\boldsymbol{Z}(f)$. For each non-invertible f in $A(X)$, both $\mathcal{Z}_A(f)$ and $\mathfrak{Z}_A(f)$ are z-filters on X and $\mathcal{Z}_A(f) \subseteq \mathfrak{Z}_A(f)$.

For an ideal I in $A(X)$ we define $\mathcal{Z}_A[I] = \bigcup\{\mathcal{Z}_A(f) : f \in I\}$ and $\mathfrak{Z}_A[I] = \bigcup\{\mathfrak{Z}_A(f) : f \in I\}$. Again $\mathcal{Z}_A[I]$ and $\mathfrak{Z}_A[I]$ are z-filters on X. The maps $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ between ideals of $A(X)$ and z-filters on X do extend the correspondences $\boldsymbol{E}$ and $\boldsymbol{Z}$ respectively to $A(X)$. In particular, for $A(X) = C^*(X)$ we have $\mathcal{Z}_A[I] = \boldsymbol{E}[I]$ (from [8, Corollary 1.3]), and for $A(X) = C(X)$ we have $\mathfrak{Z}_A[I] = \boldsymbol{Z}[I]$ (from [8, Corollary 2.4]).

The *hull* of a z-filter $\mathcal{F}$ is the set

$$h\mathcal{F} = \{\mathcal{U} \mid \mathcal{U} \text{ is a } z\text{-ultrafilter and } \mathcal{F} \subseteq \mathcal{U}\}.$$

Also, the *kernel* of a set $\mathfrak{U}$ of z-ultrafilters is

$$k\mathfrak{U} = \bigcap_{\mathcal{U} \in \mathfrak{U}} \mathcal{U}$$

(as the intersection of z-filters is a z-filter, $k\mathfrak{U}$ is a z-filter). From [8, Theorem 3.1] we have the following relationship between $\mathcal{Z}_A$ and $\mathfrak{Z}_A$:

$$\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f) \tag{2.1}$$

for each $f \in A(X)$. The following is from [9].

Lemma 2.1. *Let $A(X)$ be an intermediate ring, let $f \in A(X)$, and let $\mathcal{F}$ be a z-filter on X. Then $\mathcal{Z}_A(f) \subseteq \mathcal{F}$ if and only if $\lim_{\mathcal{F}} fh = 0$ for all $h \in A(X)$.*

For a z-filter $\mathcal{F}$ on X we define the inverse maps $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] = \{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$ and $\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}] = \{f \in A(X) : \mathfrak{Z}_A(f) \subseteq \mathcal{F}\}$. It is shown in [3] that for any intermediate ring $A(X)$, $\mathcal{Z}_A^{\leftarrow}$ maps z-filters on X to ideals in $A(X)$, and it is shown in [12, Theorem 14] that the corresponding result for $\mathfrak{Z}_A^{\leftarrow}$ holds for intermediate C-rings $A(X)$. From the definition of $\mathcal{Z}_A^{\leftarrow}$ and Lemma 2.1 we obtain the following.

Lemma 2.2. *Let $A(X)$ be an intermediate ring and let $\mathcal{F}$ be a z-filter on X. Then $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]=\{f \in A(X) : \lim_{\mathcal{F}} fh = 0$ for all $h \in A(X)\}$.*

In general, if M is a maximal ideal then $\mathfrak{Z}_A[M]$ and $\mathcal{Z}_A[M]$ need not be z-ultrafilters, but each is contained in a unique z-ultrafilter (see [3] and [8] respectively). However, if $\mathcal{U}$ is a z-ultrafilter on X, then $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ and $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ are maximal ideals in $A(X)$, as the following lemma from [8] describes.

Lemma 2.3. *Let $A(X)$ be an intermediate ring, and let $\mathcal{U}$ be a z-ultrafilter on X. Then the following hold:*

(a) *Each of* $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ *and* $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ *are maximal ideals in* $A(X)$ *and* $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}] = \mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$.

(b) *The map* $\mathcal{U} \to \mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ *defines a one-one correspondence between* z*-ultrafilters on* X *and maximal ideals in* $A(X)$.

Recall that the Stone-Čech compactification βX of X can be realized as the set of z-ultrafilters on X topologized with the hull-kernel topology, and X is naturally embedded in βX by the map $p \mapsto \mathcal{U}_p$, where $\mathcal{U}_p = \langle\{p\}\rangle$ is the principal z-ultrafilter consisting of all zero-sets containing $\{p\}$. So a point p in βX corresponds to the z-ultrafilter $\mathcal{U}_p$ on X. Throughout this paper, we identify each $p \in \beta X$ with $\mathcal{U}_p$. Then the closure of any set $U \subseteq \beta X$ is $hk(U)$, the hull of the kernel of U. By the preceding result $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p]$ is a maximal ideal which we denote by M_A^p. Thus the correspondence $p \to M_A^p$ is one-one between βX and the maximal ideals of $A(X)$.

A z-ultrafilter $\mathcal{U}$ on X is called A-*stable* if for every f in $A(X)$ there is a member of $\mathcal{U}$ on which f is bounded. For each $A(X)$ we define a subspace $\upsilon_A X$ of βX, called the A-*compactification* of X, by $\upsilon_A X = \{p \in \beta X : \mathcal{U}_p$ is A-stable$\}$ [4]. If $A(X) = C^*(X)$ then $\upsilon_A X = \beta X$ and if $A(X) = C(X)$ then then $\upsilon_A X = \upsilon X$, the Hewitt realcompactification of X. We have

$$X \subseteq \upsilon X \subseteq \upsilon_A X \subseteq \beta X.$$

It is shown in [4] that the space $\upsilon_A X$ consists of the points of βX to which every f in $A(X)$ can be continuously extended. We denote the extension of f to $\upsilon_A X$ by f^{υ_A}. It is also shown in [4] that for $f \in A(X)$ and $p \in \upsilon_A X$ we have

$$f^{\upsilon_A}(p) = \lim_{\mathcal{U}_p} f. \tag{2.2}$$

Theorem 2.4 (from [4]). *An intermediate ring* $A(X)$ *is a* C*-ring if and only if* $A(X)$ *is isomorphic to* $C(\upsilon_A X)$*, via the map* $f \mapsto f^{\upsilon_A}$.

Different intermediate rings may induce the same realcompact extension of X. For additional information on the realcompactifications of X, see [1, 4, 6]

3. The Zero-Sets of f, f^{υ_A}, and f^*

If $A(X)$ is an intermediate C-ring, $f \in A(X)$, and f^* is the Stone extension of f (to be defined below), we show the following containments:

$$cl_{\upsilon_A X}\, \boldsymbol{Z}(f) \subseteq \boldsymbol{Z}(f^{\upsilon_A}) \subseteq \boldsymbol{Z}(f^*).$$

The first containment, that $cl_{\upsilon_A X}\, \boldsymbol{Z}(f) \subseteq \boldsymbol{Z}(f^{\upsilon_A})$, is Proposition 3.2, and this containment can be strict by Example 3.3. Furthermore, equality of this containment for every $f \in A(X)$ characterizes $C(X)$ among intermediate C-rings (Theorem 3.7). The second containment, that $\boldsymbol{Z}(f^{\upsilon_A}) \subseteq \boldsymbol{Z}(f^*)$, is Proposition 3.9, and we show that equality for every $f \in A(X)$ characterizes $C^*(X)$ among intermediate rings (Theorem 3.10).

We make use of the following lemma.

Lemma 3.1. *Let $A(X)$ be an intermediate ring and $f \in A(X)$. Then $\boldsymbol{Z}(f) = \bigcap\{E : E \in \mathcal{Z}_A(f)\} = \bigcap\{E : E \in \mathfrak{Z}_A(f)\}$.*

Proof. It is proved in [10, Proposition 2.2] that $\boldsymbol{Z}(f) = \bigcap\{E : E \in \mathcal{Z}_A(f)\}$, so it is sufficient to show that $\bigcap \mathfrak{Z}_A(f) = \bigcap \mathcal{Z}_A(f)$. Since by (2.1), $\mathcal{Z}_A(f) \subseteq \mathfrak{Z}_A(f)$, it follows that $\bigcap \mathfrak{Z}_A(f) \subseteq \bigcap \mathcal{Z}_A(f)$.

For the other containment, suppose that p is a point of X such that $p \in \bigcap \mathcal{Z}_A(f)$; then $\mathcal{U}_p \supseteq \mathcal{Z}_A(f)$. Since by (2.1), $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$, we have that $\mathfrak{Z}_A(f) \subseteq \mathcal{U}_p$, and it follows that $p \in \bigcap \mathcal{U}_p \subseteq \bigcap \mathfrak{Z}_A(f)$. □

The A-stable hull of a z-filter $\mathcal{F}$ is defined as in [12]:

$$h^A(\mathcal{F}) = \{\mathcal{U}_p : \mathcal{U}_p \text{ is an } A\text{-stable } z\text{-ultrafilter on } X, \text{ and } \mathcal{F} \subseteq \mathcal{U}_p\}.$$

The closure operator in the topology of $\upsilon_A X$ is $h^A k$, because $\upsilon_A X$ is a subspace of βX (where the closure operator is hk). If E is a zero-set in X, then $kE = \langle E \rangle$, where $\langle E \rangle$ denotes the z-filter consisting of all zero-sets containing E. So

$$cl_{\upsilon_A X}\, E = h^A kE = h^A \langle E \rangle. \tag{3.1}$$

In other words $cl_{\upsilon_A X}\, E$ consists of the A-stable z-ultrafilters that contain E.

Proposition 3.2. *If $A(X)$ is an intermediate C-ring and $f \in A(X)$, then*

$$cl_{\upsilon_A X}\, \boldsymbol{Z}(f) \subseteq \boldsymbol{Z}(f^{\upsilon_A}).$$

Proof. Since $\boldsymbol{Z}(f)$ is a zero-set in X, it immediately follows from (3.1) that $cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = h^A\langle \boldsymbol{Z}(f)\rangle$. By Lemma 3.1, $\mathfrak{Z}_A(f) \subseteq \langle \boldsymbol{Z}(f)\rangle$, so $cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = h^A(\langle \boldsymbol{Z}(f)\rangle) \subseteq h^A \mathfrak{Z}_A(f)$. Also $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$, and by [12, Lemma 5], $kh\mathcal{Z}_A(f) = kh^A\mathcal{Z}_A(f)$. So $h^A\mathfrak{Z}_A(f) = h^A kh\mathcal{Z}_A(f) = h^A kh^A \mathcal{Z}_A(f) = cl_{\upsilon_A X}\, h^A\mathcal{Z}_A(f)$. It is shown in [12, Lemma 4] that $h^A\mathcal{Z}_A(f) = \boldsymbol{Z}(f^{\upsilon_A})$; so we have $cl_{\upsilon_A X}\, h^A\mathcal{Z}_A(f) = cl_{\upsilon_A X}\, \boldsymbol{Z}(f^{\upsilon_A})$. But $\boldsymbol{Z}(f^{\upsilon_A})$ is a closed set in $\upsilon_A X$, because it is a zero-set. So $cl_{\upsilon_A X}\, \boldsymbol{Z}(f^{\upsilon_A}) = \boldsymbol{Z}(f^{\upsilon_A})$. This completes the proof. □

The following example shows that in general the containment in Proposition 3.2 may be strict.

Example 3.3. Let $X = \mathbb{N}$ and $A(X) = C^*(\mathbb{N})$. Suppose E is the set of even numbers in $\mathbb{N}$, and let g be the functions whose value is zero on the set E and $1/n$ at the odd integer n. Then $cl_{\beta X}\, \boldsymbol{Z}(g)$ consists of all z-ultrafilters that contain the set E of even numbers in $\mathbb{N}$, and hence none of these z-ultrafilters contain the set of odd numbers in $\mathbb{N}$. Now for any free z-ultrafilter $\mathcal{U}_p$ containing the set of odd integers we have $\lim_{\mathcal{U}_p} g = 0$. So $p \in \boldsymbol{Z}(g^\beta)$, but clearly $p \notin cl_{\beta X}\, \boldsymbol{Z}(g)$. Thus $cl_{\beta X}\, \boldsymbol{Z}(g) \subsetneq \boldsymbol{Z}(g^\beta)$ (strict containment).

Note that this argument does not work for every function with zero-set E. Consider h which maps x to 1 if $x \notin E$ and to 0 otherwise; that is, h is the characteristic function of the complement of E. Here $cl_{\beta X}\, \boldsymbol{Z}(h)$ consists of all z-ultrafilters that contain E. As $\mathbb{N}$ is discrete, any z-ultrafilter $\mathcal{U}_p$ not in $cl_{\beta X}\, \boldsymbol{Z}(h)$ cannot contain E, and hence must contain the complement of E. But then $\lim_{\mathcal{U}_p} h = 1$ for such a z-ultrafilter $\mathcal{U}$, and hence by (2.2), $\mathcal{U}_p \notin \boldsymbol{Z}(h^\beta)$. Thus $cl_{\beta X}\, \boldsymbol{Z}(h) = \boldsymbol{Z}(h^\beta)$.

As we will see in the next lemma, there are a number of characterizations of when $cl_{\upsilon_A X}\, \boldsymbol{Z}(f)$ is equal to $\boldsymbol{Z}(f^{\upsilon_A})$. One of them is the negation of Condition ii of a theorem of Rudd [11, Theorem 1.2] that is expressed in terms of near-zero-sets. Rudd defines a subset T of X to be a near-zero-set of $f \in C(X)$ if for every $\epsilon > 0$ there exists a point $p \in T$ such that $|f(p)| < \epsilon$. The significance of T being a "near-zero-set" of f is that f is not T-regular with respect to $C^*(X)$. So in the context of intermediate rings we make the following definition.

Definition 3.4. Let $A(X)$ be an intermediate ring of continuous functions and let $f \in C(X)$. A subset T of X is called a *near-zero-set* of f with respect to $A(X)$ if f is not T-regular in $A(X)$.

It is apparent that a set E is a near-zero-set of f in the sense of Rudd [11] if and only if E is a near-zero-set of f with respect $C^*(X)$ in the sense of Definition 3.4. Also, if $B(X) \subseteq A(X)$ and if E is a near-zero-set of f with respect to $A(X)$, then E is a near-zero-set of f with respect to $B(X)$.

Lemma 3.5. *Let $A(X)$ be an intermediate ring. Then for each $f \in A(X)$, the following are equivalent:*

(a) $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$.
(b) *No near-zero-set of f with respect to $A(X)$ is completely separated from $\boldsymbol{Z}(f)$.*

Proof. $(b) \Rightarrow (a)$. Suppose that $\mathfrak{Z}_A(f) \neq \langle \boldsymbol{Z}(f) \rangle$. Then by Lemma 3.1 there exists a zero-set E such that $\boldsymbol{Z}(f) \subseteq E$ and $E \notin \mathfrak{Z}_A(f)$. By the definition of $\mathfrak{Z}_A(f)$ it follows that there exists a zero-set H in the complement of E for which f is not H-regular with respect to $A(X)$. So H is a near-zero-set of f with respect to $A(X)$. But $\boldsymbol{Z}(f)$ and H are disjoint zero-sets in X and hence are completely separated ([7, page 17]).

$(a) \Rightarrow (b)$. Suppose that there exists a near-zero-set T of f with respect to $A(X)$ which is completely separated from $\boldsymbol{Z}(f)$. So there is a zero-set H disjoint from $\boldsymbol{Z}(f)$ such that $T \subseteq H$. Since f is not T-regular, it follows that f is not H-regular either. So H is a zero-set in the complement of $\boldsymbol{Z}(f)$ and f is not H-regular in $A(X)$. It follows from the definition of $\mathfrak{Z}_A(f)$ that $\boldsymbol{Z}(f) \notin \mathfrak{Z}_A(f)$, and so $\mathfrak{Z}_A(f) \neq \langle \boldsymbol{Z}(f) \rangle$. □

In the case of C-rings, we have the following lemma.

Lemma 3.6. *Let $A(X)$ be an intermediate C-ring. Then for each $f \in A(X)$, the following are equivalent:*

(a) $cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^{\upsilon_A})$.
(b) $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$.

Proof. $(a) \Rightarrow (b)$. Suppose $cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^{\upsilon_A})$. By Lemma 3.1, $\mathfrak{Z}_A(f) \subseteq \langle \boldsymbol{Z}(f) \rangle$. For the other inclusion, suppose E is a zero-set such that $E \supseteq \boldsymbol{Z}(f)$. Then $cl_{\upsilon_A X}\, E \supseteq cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^{\upsilon_A})$ by hypothesis. But by [12, Theorem 6], $\mathfrak{Z}_A(f) = \{E \in \boldsymbol{Z}(X) \mid \boldsymbol{Z}(f^{\upsilon_A}) \subseteq cl_{\upsilon_A} E\}$. Hence we have $E \in \mathfrak{Z}_A(f)$.

$(b) \Rightarrow (a)$. Suppose that $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$. Then

$$\begin{aligned} cl_{\beta X}\, \boldsymbol{Z}(f^{\upsilon_A}) &= cl_{\beta X}\, h^A \mathcal{Z}_A(f) && \text{(by [12, Lemma 4])} \\ &= hkh^A \mathcal{Z}_A(f) && \text{(by definition of closure in } \beta X) \\ &= hkh\mathcal{Z}_A(f) && \text{(by [12, Lemma 5])} \\ &= h\mathfrak{Z}_A(f) && \text{(by (2.1))} \\ &= h\langle \boldsymbol{Z}(f) \rangle && \text{(by hypothesis)} \end{aligned}$$

Now, since $\boldsymbol{Z}(f^{\upsilon_A})$ is closed in $\upsilon_A X$, we have

$$\begin{aligned} \boldsymbol{Z}(f^{\upsilon_A}) &= cl_{\upsilon_A X}\, \boldsymbol{Z}(f^{\upsilon_A}) = cl_{\beta X}\, \boldsymbol{Z}(f^{\upsilon_A}) \cap \upsilon_A(X) \\ &= h\langle \boldsymbol{Z}(f) \rangle \cap \upsilon_A(X) = h^A \langle \boldsymbol{Z}(f) \rangle = cl_{\upsilon_A X}\, \boldsymbol{Z}(f). \end{aligned}$$

□

As a special case of Proposition 3.2, we have that $cl_{\beta X}\, \boldsymbol{Z}(f) \subseteq \boldsymbol{Z}(f^\beta)$ and we know from Example 3.3 that the containment can be strict. But in general $cl_{\upsilon X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^\upsilon)$, where υX is the Hewitt realcompactification of X (see [7, §8.8(b)]). The next theorem shows that this latter property characterizes $C(X)$ among intermediate C-rings on X.

Theorem 3.7. *Let $A(X)$ be an intermediate C-ring. Then $A(X) = C(X)$ if and only if every $f \in A(X)$ satisfies $cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^{\upsilon_A})$.*

Proof. The desired result is immediate from the following equivalences:

$$\begin{aligned} & A(X) = C(X) \\ \Leftrightarrow\ & \mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle \text{ for each } f \in A(X) && \text{(by [8, Theorem 2.3])} \\ \Leftrightarrow\ & cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^{\upsilon_A}) \text{ for each } f \in A(X) && \text{(by Lemma 3.6).} \end{aligned}$$ □

Swardson [14, Corollary 4.3] shows that a completely regular space X satisfies $cl_{\beta X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^\beta)$ for every $f \in C^*(X)$ if and only if X is pseudocompact. We show that this characterization follows from Theorem 3.7 above. First, recall that a space X is pseudocompact if every continuous real-valued function on X is bounded. So X is pseudocompact if and only if $C(X) = C^*(X)$.

Corollary 3.8. [14, Corollary 4.3] *Let X be a completely regular space. Then $cl_{\beta X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^\beta)$ for every $f \in C^*(X)$ if and only if X is pseudocompact.*

Proof. Suppose that $cl_{\beta X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^\beta)$ for every $f \in C^*(X)$. Setting $A(X) = C^*(X)$, we have that $\upsilon_A X = \beta X$, and hence $cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^{\upsilon_A})$, for every $f \in A(X)$. It follow by Theorem 3.5 that $A(X) = C(X)$. But this means that $C^*(X) = C(X)$, and so X is pseudocompact.

Conversely, suppose that X is pseudocompact and set $A(X) = C^*(X)$. As $C^*(X) = C(X)$, we also have $A(X) = C(X)$, and it follows by Theorem 3.7 that $cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^{\upsilon_A})$. But since $A(X) = C^*(X)$ we have that $\upsilon_A X = \beta X$, and it follows that $cl_{\beta X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^\beta)$ for every $f \in C^*(X)$. □

For $f \in C(X)$, the function f^* denotes the Stone extension of f to the one-point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. This means that $f^* : \beta X \to \mathbb{R}^*$ extends the function f^{υ_A}, and that(2.2) extends as follows: every point $p \in \beta X$ corresponds to a z-ultrafilter $\mathcal{U}_p$ on X, and the value of f^* at p is

$$f^*(p) = \lim_{\mathcal{U}_p} f. \tag{3.2}$$

We observe that for any intermediate ring $A(X)$, the zero-set of f^{υ_A} is contained in the zero-set of f^*, as follows.

Proposition 3.9. *If $A(X)$ is an intermediate ring and $f \in A(X)$, then*

$$\boldsymbol{Z}(f^{\upsilon_A}) \subseteq \boldsymbol{Z}(f^*).$$

Proof. By (3.2), the zero-set of f^* is $\boldsymbol{Z}(f^*) = \{p \in \beta X : \lim_{\mathcal{U}_p} f = 0\}$. It follows by Lemma 2.1 that a z-ultrafilter $\mathcal{U}_p$ that contains $\mathcal{Z}_A(f)$ must satisfy $\lim_{\mathcal{U}_p} fh = 0$ for every $h \in A(X)$, and in particular for $h = 1$. Hence $\mathcal{U}_p$ belongs to $\boldsymbol{Z}(f^*)$, that is $h\mathcal{Z}_A(f) \subseteq \boldsymbol{Z}(f^*)$. On the other hand, by [12, Lemma 4] we have $\boldsymbol{Z}(f^{\upsilon_A}) = h^A \mathcal{Z}_A(f)$. Thus $\boldsymbol{Z}(f^{\upsilon_A}) = h^A \mathcal{Z}_A(f) \subseteq h\mathcal{Z}_A(f) \subseteq \boldsymbol{Z}(f^*)$. □

Theorem 3.10. *Let $A(X)$ be an intermediate ring. Then $A(X) = C^*(X)$ if and only if every $f \in A(X)$ satisfies $\boldsymbol{Z}(f^{\upsilon_A}) = \boldsymbol{Z}(f^*)$.*

Proof. If $A(X) = C^*(X)$ then $\upsilon_A X = \beta X$ and $f^{\upsilon_A} = f^\beta$. Also, as f is bounded, f^* never attains the value ∞, and hence $f^{\upsilon_A} = f^*$. So $\boldsymbol{Z}(f^{\upsilon_A}) = \boldsymbol{Z}(f^*)$ for every $f \in C^*(X)$.

Conversely, suppose that $A(X) \neq C^*(X)$. Then there is an unbounded function $g \in A(X)$. Without loss of generality assume that $g > 0$, and let $f = 1/g$. Let $\mathcal{U}$ be any z-ultrafilter containing the zero-sets $\{x \in X : g(x) \geq n\}$. Clearly $\lim_{\mathcal{U}} f = 0$, so $\mathcal{U} \in \boldsymbol{Z}(f^*)$. But $\mathcal{U}$ is not A-stable because f is unbounded on every set in $\mathcal{U}$, so $\mathcal{U} \notin \boldsymbol{Z}(f^{\upsilon_A})$. □

Rudd [11] considers the question of when $\boldsymbol{Z}(f^*) = cl_{\beta X}\, \boldsymbol{Z}(f)$. We give a short proof of the equivalence of the first two parts of [11, Theorem 1.2] by using the results of this article.

Theorem 3.11. *Let $f \in C(X)$. Then $\boldsymbol{Z}(f^*) \neq cl_{\beta X}\, \boldsymbol{Z}(f)$ if and only if there is a near-zero-set of f with respect to $C^*(X)$ which is completely separated from $\boldsymbol{Z}(f)$.*

Proof. First, note that two sets in X are completely separated if and only if they are contained in disjoint zero-sets [7, §1.15].

($\Longrightarrow$) Suppose that $\boldsymbol{Z}(f^*) \neq cl_{\beta X}\, \boldsymbol{Z}(f)$. Then there is a z-ultrafilter $\mathcal{U}_p \in \boldsymbol{Z}(f^*)$ but $\mathcal{U}_p \notin cl_{\beta X}\, \boldsymbol{Z}(f)$. This means that $\boldsymbol{Z}(f) \notin \mathcal{U}_p$. But then there is a zero-set E in $\mathcal{U}_p$ which does not meet $\boldsymbol{Z}(f)$, and hence by [7, §1.15], E is completely separated from $\boldsymbol{Z}(f)$. But since $E \in \mathcal{U}_p$ and $\lim_{\mathcal{U}_p} f = 0$, it follows that E is a near-zero-set of f with respect to $C^*(X)$.

($\Longleftarrow$) Suppose that E is a near-zero-set of f with respect to $C^*(X)$ and suppose that E is completely separated from $\boldsymbol{Z}(f)$. Then by [7, §1.15], there is a zero-set F containing E such that F and $\boldsymbol{Z}(f)$ are disjoint. As the property of being a near-zero-set of a function with respect to $A(X)$ is closed under supersets, F is a near-zero-set of f with respect to $C^*(X)$. Since F is a near-zeros-set of f with respect to $C^*(X)$, the function f achieves values on F arbitrarily close to zero. Thus the sets $\{x : |f(x)| \leq 1/n\} \cap F$, for $n \in \mathbb{N}$, are nonempty zero-sets with the finite intersection property. So there is a z-ultrafilter $\mathcal{U}_p$ containing these sets. Clearly $\lim_{\mathcal{U}_p} f = 0$, so $p \in \boldsymbol{Z}(f^*)$. On the other hand, since $\boldsymbol{Z}(f)$ and F are completely separated, $\boldsymbol{Z}(f) \notin \mathcal{U}_p$, and so $p \notin cl_{\beta X}\, \boldsymbol{Z}(f)$. □

4. Closures of Zero-Sets of X in $\upsilon_A X$

We now consider the problem of when the closure of a zero-set in X is a zero-set in $\upsilon_A X$. It was shown by Rudd in [11, Example 1.5] that if $E \in \boldsymbol{Z}[X]$, then it is possible that $cl_{\beta X}\, E \notin \boldsymbol{Z}[\beta X]$. Hence it is not always the case that the closure of a zero-set in X is a zero-set in $\upsilon_A(X)$. The following example is an alternative to Rudd's example that fits well with the formalism of this paper.

Example 4.1. Let $A(X) = C^*([0,\infty))$ and let $f \in A(X)$ be any function whose zero-set is $E = \{1, 2, \ldots\}$. So $cl_{\beta X}\, \boldsymbol{Z}(f)$ consists of those z-ultrafilters containing E. Choose a sequence $F = \{a_n : n = 1, 2, \ldots\}$ of positive numbers disjoint from E such that $|a_n - n| < 1$ for each n and $f(a_n) \to 0$ as $n \to \infty$. Since the a_n are close to n, the set F is unbounded, and hence F is contained in some free z-ultrafilter $\mathcal{U}_p$. As $\mathcal{U}_p$ is free, $\lim_{\mathcal{U}_p} f = 0$. Hence by (2.2), $p \in \boldsymbol{Z}(f^\beta)$. But $p \notin cl_{\beta X}\, \boldsymbol{Z}(f)$ because $E \cap F = \emptyset$. So $cl_{\beta X}\, \boldsymbol{Z}(f) \subsetneq \boldsymbol{Z}(f^\beta)$ (strict containment). Since this is true for each f with zero-set E, it follows that $cl_{\beta X}\, E$ is not a zero-set in βX. (Note that if there were a function $g \in C(\beta X)$ whose zero-set were $cl_{\beta X}\, E$, then the restriction of g to X would have zero-set E.)

In [14, §5], Swardson asked the following two questions. When is $cl_{\beta X}\, E$ a zero-set in βX if E is a zero-set of X? When does $cl_{\beta X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^\beta)$ for $f \in C^*(X)$? Some progress to answering these had already been given. For example in [15, Lemma 5], Terada found a characterization for when $cl_{\beta X}\, \boldsymbol{Z}(f)$ is a zero-set of βX. For other results on the closures of zero-sets in βX see [5]. We generalize these questions to $A(X)$, and compare the following two statements.

(1) For every zero-set E in X, its closure $cl_{\upsilon_A X}\, E$ is a zero-set in $\upsilon_A X$.
(2) For every $f \in A(X)$ we have $cl_{\upsilon_A X}\, \boldsymbol{Z}(f) = \boldsymbol{Z}(f^{\upsilon_A})$.

Let E be a zero-set in X. Statement (2) implies that for every $f \in A(X)$ with $\boldsymbol{Z}(f) = E$ we have $cl_{\upsilon_A X}\, \boldsymbol{E} = \boldsymbol{Z}(f^{\upsilon_A})$ whereas Statement (1) follows if this equality holds only for some $f \in A(X)$ (so that $cl_{\upsilon_A X}\, \boldsymbol{E}$ is a zero-set in $\upsilon_A X$). So Statement (2) clearly implies Statement (1). We have shown (Theorem 3.7) that the second statement characterizes $C(X)$ among intermediate C-rings. We now show that the first statement characterizes those C-rings $A(X)$ for which each z-filter is a $\mathfrak{Z}_A$-filter. By a $\mathfrak{Z}_A$-*filter* we mean a z-filter $\mathcal{F}$ with the property that

$$\mathfrak{Z}_A \mathfrak{Z}_A^{\leftarrow}[\mathcal{F}] = \mathcal{F}.$$

Note that if $A(X) = C(X)$ then every z-filter is a $\mathfrak{Z}_A$-filter because in this case $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$ for each f, and it is known that $\boldsymbol{Z}\boldsymbol{Z}^{\leftarrow}[\mathcal{F}] = \mathcal{F}$ for every z-filter $\mathcal{F}$ ([7, p. 26]).

Theorem 4.2. *Let $A(X)$ be an intermediate C-ring. The following are equivalent.*

(a) *Every z-filter on X is a $\mathfrak{Z}_A$-filter.*

(b) *For every zero-set E in X, there exists $f \in A(X)$ such that $E = \boldsymbol{Z}(f)$ and $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$.*
(c) *For every zero-set E in X, there exists $f \in A(X)$ such that $E = \boldsymbol{Z}(f)$ and no near-zero-set of f with respect to $A(X)$ is completely separated from $\boldsymbol{Z}(f)$.*
(d) *For every zero-set E in X, its closure $cl_{\upsilon_A X} E$ is a zero-set in $\upsilon_A X$.*

Proof. $(a) \Rightarrow (b)$. Let E be a zero-set in X, let $\mathcal{F} = \langle E \rangle$, and let $I = \mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]$. By hypothesis, $\mathfrak{Z}_A[I] = \mathcal{F}$. So there exists $f \in I$ such that $E \in \mathfrak{Z}_A(f)$. Then $\langle E \rangle \subseteq \mathfrak{Z}_A(f) \subseteq \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]] = \mathcal{F} = \langle E \rangle$; hence $\mathfrak{Z}_A(f) = \langle E \rangle$. Thus by Lemma 3.1, we have $\boldsymbol{Z}(f) = E$, and so $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$.

$(b) \Rightarrow (a)$. Let $\mathcal{F}$ be a z-filter on X. Clearly $\mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]] \subseteq \mathcal{F}$. Suppose $E \in \mathcal{F}$. Then by hypothesis there exists $f \in A(X)$ such that $E = \boldsymbol{Z}(f)$ and $\mathfrak{Z}_A(f) = \langle E \rangle$. Thus $\mathfrak{Z}_A(f) \subseteq \mathcal{F}$ and so $f \in \mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]$. This means that $E \in \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]]$. Thus $\mathcal{F} \subseteq \mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]]$.

$(b) \Leftrightarrow (c)$. This is immediate from Lemma 3.5.

$(b) \Rightarrow (d)$. Let E be a zero-set in X and let $f \in A(X)$ such that $\boldsymbol{Z}(f) = E$ and $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$. Then, by Lemma 3.6, $cl_{\upsilon_A X} \boldsymbol{Z}(f) = \boldsymbol{Z}(f^{\upsilon_A})$, and so $cl_{\upsilon_A X} \boldsymbol{Z}(f) = cl_{\upsilon_A X} E$ is a zero-set in $\upsilon_A X$.

$(d) \Rightarrow (b)$. Let E be a zero-set in X with the property that $cl_{\upsilon_A X} E$ is a zero-set in $\upsilon_A X$. Let $\tilde{f} \in C(\upsilon_A X)$, such that $\boldsymbol{Z}(\tilde{f}) = cl_{\upsilon_A X} E$. Since $A(X)$ is a C-ring, by Theorem 2.4, $A(X)$ is isomorphic to $C(\upsilon_A(X))$ by the isomorphism $f \mapsto f^{\upsilon_A}$. Thus there exists f, such that $f^{\upsilon_A} = \tilde{f}$. It follows that $\boldsymbol{Z}(f) = E$ and $\boldsymbol{Z}(f^{\upsilon_A}) = cl_{\upsilon_A X} \boldsymbol{Z}(f)$. It follows by Lemma 3.6 that for this f we have $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$. □

5. Maximal Ideals under $\mathfrak{Z}_A$

Recall that $\mathfrak{Z}_A$ not only maps ideals in $A(X)$ to z-filters on X, but that $\mathfrak{Z}_A$ extends the map $\boldsymbol{Z}$ defined for $C(X)$ to all intermediate rings $A(X)$. This means that if $A(X) = C(X)$ then $\mathfrak{Z}_A[I] = \boldsymbol{Z}[I]$ for every ideal I in $A(X)$. It is known that $\boldsymbol{Z}$ maps each maximal ideal of $C(X)$ to a z-ultrafilter on X. We investigate the corresponding property for $\mathfrak{Z}_A$ on any intermediate ring $A(X)$. It was shown in [8, Example 4.5] that if M is a maximal ideal in $A(X)$ it does not necessarily follow that $\mathfrak{Z}_A[M]$ is a z-ultrafilter on X. This raises the following question: does the property of $\mathfrak{Z}_A$ mapping maximal ideals to z-ultrafilters characterize $C(X)$ among intermediate rings $A(X)$? We show that the answer to this question is negative by exhibiting intermediate rings $A(X)$, different from $C(X)$, for which the property does hold (Theorem 5.2). We begin by giving a sufficient condition for an intermediate ring $A(X)$ to have this property.

Theorem 5.1. *Let $A(X)$ be an intermediate C-ring for which every z-filter on X is a $\mathfrak{Z}_A$-filter. If M is a maximal ideal in $A(X)$, then $\mathfrak{Z}_A[M]$ is a z-ultrafilter on X.*

Proof. By Lemma 2.3(b), there is a unique z-ultrafilter $\mathcal{U}$ such that $\mathfrak{Z}_A[M] \subseteq \mathcal{U}$. Now, let $E \in \mathcal{U}$. Then by Theorem 4.2 (the fact that (a) implies (b)), there exists $f \in A(X)$ such that $\mathfrak{Z}_A(f) = \langle E \rangle \subseteq \mathcal{U}$. By definition of $\mathfrak{Z}_A^{\leftarrow}$, it holds that $f \in M$. Hence $E \in \mathfrak{Z}_A[M]$. □

A P-space is a completely regular Hausdorff space in which every zero-set is open. We show that if $A(X)$ is an intermediate ring on a P-space space X, then $\mathfrak{Z}_A$ does map maximal ideals to z-ultrafilters. In particular, if X is a P-space, then $C^*(X)$ has this property, and so the property does not characterize $C(X)$ among intermediate rings.

Theorem 5.2. *If $A(X)$ is an intermediate C-ring on a P-space X and if M is a maximal ideal in $A(X)$, then $\mathfrak{Z}_A[M]$ is a z-ultrafilter on X.*

Proof. Because X is a P-space, for every zero-set E in X, it holds that E is open, and hence the characteristic function f of the complement of E is continuous (and hence in $A(X)$). By definition of f, we have that $\boldsymbol{Z}(f) = E$, and it is easy to see from the definition of $\mathfrak{Z}_A$ that $\mathfrak{Z}_A(f) = \langle E \rangle$. Then by Theorem 4.2 (the fact that (b) implies (a)), every z-filter on X is a $\mathfrak{Z}_A$-filter. Then by Theorem 5.1, if M is a maximal ideal in $A(X)$, then $\mathfrak{Z}_A[M]$ is a z-ultrafilter on X. □

The following problem remains open.

Problem 5.3. *Characterize those intermediate rings $A(X)$ for which $\mathfrak{Z}_A[M]$ is a z-ultrafilters on X whenever M is a maximal ideal in $A(X)$.*

Any characterization of the intermediate rings satisfying the condition in the problem must involve properties of both the topology of X and the ring structure of $A(X)$. This is because from Theorem 5.2 we know that if X is a P-space, then the condition holds for any intermediate ring $A(X)$, whereas it is well known that if $A(X) = C(X)$ then the condition holds for any completely regular topological space X.

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