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Correspondences between ideals and z-filters for rings of continuous functions between C^* and C

Abstract. Let X be a completely regular topological space. Let $A(X)$ be a ring of continuous functions between $C^*(X)$ and $C(X)$, that is $C^*(X) \subseteq A(X) \subseteq C(X)$. In [9], a correspondence $\mathcal{Z}_A$ between ideals of $A(X)$ and z-filters on X is defined. Here we show that $\mathcal{Z}_A$ extends the well-known correspondence for $C^*(X)$ to all rings $A(X)$. We define a new correspondence $\mathfrak{Z}_A$ and show that it extends the well-known correspondence for $C(X)$ to all rings $A(X)$. We give a formula that relates the two correspondences. We use properties of $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ to characterize $C^*(X)$ and $C(X)$ among all rings $A(X)$. We show that $\mathfrak{Z}_A$ defines a one-one correspondence between maximal ideals in $A(X)$ and the z-ultrafilters in X.

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Let X be a completely regular topological space, $C(X)$ the ring of all continuous real-valued functions on X, and $C^*(X)$ the ring of bounded continuous real-valued functions on X. Let $A(X)$ be any ring of real-valued continuous functions between $C^*(X)$ and $C(X)$, that is

$$C^*(X) \subseteq A(X) \subseteq C(X).$$

This paper is concerned with correspondences between ideals in $A(X)$ and z-filters on X.

There is a well-known correspondence $\boldsymbol{E}$ between ideals in $C^*(X)$ and z-filters in X [6, Problem 2L]. However, $\boldsymbol{E}$ does not associate filters to ideals when applied to rings of continuous function that strictly contain $C^*(X)$. There is also a well-known correspondence $\boldsymbol{Z}$ between ideals in $C(X)$ and z-filters in X that maps each function to its zero set. However, although $\boldsymbol{Z}$ is defined on all of $C(X)$, it does not associate filters to ideals when applied to rings of continuous functions strictly contained in $C(X)$ [6, Section 2.4]. It is known that the correspondence $\boldsymbol{Z}$ for $C(X)$ distinguishes between ideals of $C(X)$ much more sharply than the correspondence $\boldsymbol{E}$ does for ideals of $C^*(X)$. Indeed, in [6, p. 30], the correspondence for $C^*(X)$ is called "rudimentary." Of course, the correspondences $\boldsymbol{Z}$ and $\boldsymbol{E}$ cannot be compared directly because they are defined for different rings.

In a previous article [9], we defined the map $\mathcal{Z}_A$ which associates a z-filter on X to each ideal I of any given ring of continuous functions $A(X)$ with $C^*(X) \subseteq A(X) \subseteq C(X)$. In this paper we describe a new map $\mathfrak{Z}_A$ which also associates a z-filter on X to each ideal I of $A(X)$. We show that the correspondences $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ extend the well-known correspondences for $C^*(X)$ and $C(X)$, respectively, to all such rings $A(X)$ (Corollaries 1.3 and 2.4). We show that the correspondences $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ characterize $C^*(X)$ and $C(X)$ among all rings $A(X)$ between $C^*(X)$ and $C(X)$ (Theorems 1.2 and 2.3). The maps $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ also allow us to compare the relationship between the correspondences defined separately for $C^*(X)$ and $C(X)$ by comparing their extensions to all such rings $A(X)$. We give an explicit formula that defines this relationship (Theorem 3.1). We also show that $\mathfrak{Z}_A$ is a one-one correspondence between maximal ideals in $A(X)$ and z-ultrafilters on X (Theorem 4.8).

Rings of continuous functions between $C^*(X)$ and $C(X)$ have been studied by several authors. D. Plank [8] gives a description of their maximal ideals. In [9], [10], [2], and [5] the map $\mathcal{Z}_A$ has been used as a correspondence between ideals in $A(X)$ and z-filters on X. See [3] for an extended class of rings and [4] for a relationship between subrings of $C(X)$ containing $C^*(X)$ and real compactifications. Also, work relating such rings to lattice structures is in [1] and [7].

1. $C^*(X)$ and the map $\mathcal{Z}_A$.

In this section we show that the correspondence $\mathcal{Z}_A$, defined in [9], between ideals in subrings $A(X)$ of $C(X)$ containing $C^*(X)$ and z-filters on X extends the natural correspondence for $C^*(X)$, and we then prove additional properties about the correspondence $\mathcal{Z}_A$. Recall that the correspondence for $C^*(X)$ is described as follows. For $f \in C^*(X)$ let $\boldsymbol{E}_\epsilon(f) = \{x : |f(x)| \leq \epsilon\}$, and let $\boldsymbol{E}(f) = \{\boldsymbol{E}_\epsilon(f) : \epsilon > 0\}$; then the correspondence is given by

$$I \to \boldsymbol{E}[I] = \cup\{\boldsymbol{E}(f) : f \in I\}.$$

It is shown in [9] that for a ring $A(X)$ between $C^*(X)$ and $C(X)$, the map $\mathcal{Z}_A$ associates a z-filter on X to each non-invertible function $f \in A(X)$ as follows. If E is a subset of X then f is *E-regular in $A(X)$* if there exists $g \in A(X)$ such that $fg(x) = 1$ for $x \in E$. When $A(X)$ is understood by context, we simply say that f is E-regular, without explicit reference to the ring. Then

$$\mathcal{Z}_A(f) = \{E \in \boldsymbol{Z}[X] : f \text{ is } E^c\text{-regular}\},$$

where $\boldsymbol{Z}[X]$ is the family of zero sets of X. Given a set $S \subseteq A(X)$, we define $\mathcal{Z}_A[S] = \cup\{\mathcal{Z}_A(f) : f \in S\}$. It is shown in [9] that the map $I \to \mathcal{Z}_A[I]$ is a correspondence between ideals I in $A(X)$ and z-filters on X. We now show that the correspondence $\mathcal{Z}_A$ extends the correspondence $\boldsymbol{E}$ for $C^*(X)$ to all subrings of $C(X)$ containing $C^*(X)$. Indeed, the next theorem shows that $\mathcal{Z}_A$ characterizes $C^*(X)$ among subrings of $C(X)$ containing $C^*(X)$. We make use of the following lemma about $\mathcal{Z}_A$ proved in [9], which we state here for convenience.

LEMMA 1.1 *Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$, let $f, g \in A(X)$, and let $\mathcal{F}$ be a z-filter on X. Then*

(a) *f is noninvertible in $A(X)$ if and only if $\mathcal{Z}_A(f)$ is a z-filter on X.*

(b) *$\mathcal{Z}_A(f) \subseteq \mathcal{F}$ if and only if $\lim_{\mathcal{F}} fh = 0$ for all $h \in A(X)$.*

We now give a characterization of $C^*(X)$ among subrings of $C(X)$ containing $C^*(X)$ in terms of the relationship between $\mathcal{Z}_A$ and $\boldsymbol{E}$. As usual, $f \vee g$ and $f \wedge g$ are pointwise maxima and minima respectively (as in [6]). If $\mathcal{H}$ is a collection of zero-sets in X we use the notation $\langle \mathcal{H} \rangle$ as follows. If $\mathcal{H}$ is a z-filter base then $\langle \mathcal{H} \rangle$ is the z-filter generated by $\mathcal{H}$. Otherwise $\langle \mathcal{H} \rangle = \boldsymbol{Z}[X]$, the collection of all zero-sets in X.

THEOREM 1.2 *Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$. Then $A(X) = C^*(X)$ if and only if $\mathcal{Z}_A(f) = \langle \boldsymbol{E}(f) \rangle$ for all $f \in A(X)$.*

PROOF Suppose $A(X) = C^*(X)$. Let $f \in A(X)$. If f is invertible in $A(X)$ then clearly $\mathcal{Z}_A(f) = Z[X] = \langle \boldsymbol{E}(f) \rangle$. If f is noninvertible in $A(X)$, let $E \in \boldsymbol{E}(f)$. Then for some $\epsilon > 0$, $|f(x)| > \epsilon$ for all $x \in E^c$. Let $h = f \vee \epsilon$. Clearly $h^{-1} \in C^*(X)$ and $h^{-1}f(x) = 1$ for $x \in E^c$, and hence $E \in \mathcal{Z}_A(f)$. For the other containment, suppose $E \in \mathcal{Z}_A(f)$. Then there exists $g \in A(X)$ such that $fg(x) = 1$ for $x \in E^c$. Since g is bounded there exists $\epsilon > 0$ such that $|f(x)| > \epsilon$ for all $x \in E^c$. Thus $E \supseteq \{x : |f(x)| \leq \epsilon\}$, so $E \in \langle \boldsymbol{E}(f) \rangle$.

Conversely, suppose $\mathcal{Z}_A(f) = \langle \boldsymbol{E}(f) \rangle$ for all $f \in A(X)$. If there is an unbounded function $f \in A(X)$ then $h = 1/(f^2 + 1)$ is bounded and never zero. By definition $\langle \boldsymbol{E}(h) \rangle$ is a z-filter, so $\mathcal{Z}_A(h)$ is a z-filter. But h is invertible in $A(X)$, contradicting Lemma 1.1(a). ■

COROLLARY 1.3 *For any ideal $I \in C^*(X)$, $\mathcal{Z}_{C^*}[I] = \boldsymbol{E}[I]$.*

PROOF It follows from Theorem 1.2 and the definitions that $\mathcal{Z}_{C^*}[I] = \bigcup_{f \in I} \langle \boldsymbol{E}(f) \rangle$. It remains to show that $\bigcup_{f \in I} \langle \boldsymbol{E}(f) \rangle = \bigcup_{f \in I} \boldsymbol{E}(f)$. The right to left containment follows directly from the definitions. The left to right containment also makes use of the fact that $\boldsymbol{E}[I]$ is a z-filter and hence upward closed [6, p. 33]. ■

For the remainder of this section, we prove more properties about the map $\mathcal{Z}_A$. First we show how to characterize in terms of $\mathcal{Z}_A$ which sets are H-regular.

LEMMA 1.4 *Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$, let $f \in A(X)$, and let H be a zero set in X. Then $H \cap F \neq \varnothing$ for every $F \in \mathcal{Z}_A(f)$ if and only if f is not H-regular. In particular, f is not F-regular for any $F \in \mathcal{Z}_A(f)$.*

PROOF Suppose H is disjoint from some $F \in \mathcal{Z}_A(f)$. Then $H \subset F^c$. By the definition of $\mathcal{Z}_A(f)$, f is F^c-regular. But then f is H-regular. For the other implication, suppose H meets every $F \in \mathcal{Z}_A(f)$. Then there is a z-filter $\mathcal{F}$ containing H and $\mathcal{Z}_A(f)$. Now if f is H-regular then there exists $h \in A(X)$ such that $fh(x) = 1$ for $x \in H$, and in this case $\lim_{\mathcal{F}} fh \neq 0$. But by Lemma 1.1(b), $\lim_{\mathcal{F}} fh = 0$ for all $h \in A(X)$. It follows that f is not H-regular. ■

The next lemma shows to what extent $\mathcal{Z}_A$ maps products and sums of functions to respectively meets and joins on the lattice of z-filters.

LEMMA 1.5 *Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$, and let $f, g \in A(X)$.*

(a) $\mathcal{Z}_A(fg) = \mathcal{Z}_A(f) \wedge \mathcal{Z}_A(g)$

(b) $\mathcal{Z}_A(f+g) \subseteq \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$

(c) If $f, g \geq 0$, then $\mathcal{Z}_A(f+g) = \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$

PROOF (a) The containment $\mathcal{Z}_A(fg) \subseteq \mathcal{Z}_A(f) \wedge \mathcal{Z}_A(g)$ follows from the fact that if fg is locally invertible on E^c, then so are f and g. For the other containment, let $E \in \mathcal{Z}_A(f) \wedge \mathcal{Z}_A(g)$. Then there exist $h, k \in A(X)$ such that $fh(x) = 1$ and $gk(x) = 1$ for $x \in E^c$. Then $fghk(x) = 1$ for $x \in E^c$, and so $E \in \mathcal{Z}_A(fg)$.

(b) Let $\mathcal{F} = \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$. Then by Lemma 1.1(b), $\lim_{\mathcal{F}} fh = 0$ and $\lim_{\mathcal{F}} gh = 0$ for all $h \in A(X)$. Thus $\lim_{\mathcal{F}}(f+g)h = \lim_{\mathcal{F}} fh + \lim_{\mathcal{F}} gh = 0$ for all $h \in A(X)$, and so by Lemma 1.1(b), $\mathcal{Z}_A(f+g) \subseteq \mathcal{F}$.

(c) Since $0 < f \leq f+g$, it follows from Lemma 1(d) of [9] that $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(f+g)$. Similarly, $\mathcal{Z}_A(g) \subseteq \mathcal{Z}_A(f+g)$, and so $\mathcal{Z}_A(f) \vee \mathcal{Z}_A(g) \subseteq \mathcal{Z}_A(f+g)$. Equality then follows from (b). ■

Notice that Lemma 1.5 (a) implies that for any $f \in A(X)$ we have $\mathcal{Z}_A(f) = \mathcal{Z}_A(f^2)$. This also follows directly from the definition of $\mathcal{Z}_A(f)$. Furthermore, the opposite containment of (b) does not in general hold, for $f = x$ and $g = -x$ is a counterexample with $A(X) = C(X)$.

2. $C(X)$ and the map $\mathfrak{Z}_A$.

In this section we define a new correspondence $\mathfrak{Z}_A$ between ideals and z-filters for a rings $A(X)$ between $C^*(X)$ and $C(X)$, and show that it extends the well-known correspondence for $C(X)$. Recall that the natural correspondence between ideals in $C(X)$ and z-filters on X is described as follows. To each $f \in C(X)$ we associate its zero set $\boldsymbol{Z}(f)$, and to an ideal I in $C(X)$, the correspondence is defined as follows:

$$I \to \boldsymbol{Z}[I] = \cup\{\boldsymbol{Z}(f) : f \in I\}.$$

For rings $A(X)$ of between $C^*(X)$ and $C(X)$, we define a map $\mathfrak{Z}_A$ as follows.

DEFINITION 2.1 Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$. For $f \in A(X)$ we set

$$\mathfrak{Z}_A(f) = \{E \in \boldsymbol{Z}[X] : \text{for all zero sets } H \subset E^c,\ f \text{ is } H\text{-regular}\}.$$

Given a set $S \subseteq A(X)$, we define $\mathfrak{Z}_A[S] = \cup\{\mathfrak{Z}_A(f) : f \in S\}$.

We now show that the correspondence $\mathfrak{Z}_A$ extends the correspondence $\boldsymbol{Z}$ for $C(X)$ to all subrings $A(X)$ of $C(X)$ containing $C^*(X)$. Indeed, the next theorem shows that $\mathfrak{Z}_A$ characterizes $C(X)$ among all such rings $A(X)$. We first prove the following fundamental proposition.

Proposition 2.2 *Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$, and let $f \in A(X)$. Then f is not invertible in $A(X)$ if and only if $\mathfrak{Z}_A(f)$ is a z-filter on X. When f is invertible in $A(X)$, $\mathfrak{Z}_A(f) = \boldsymbol{Z}[X]$, the collection of all zero sets on X.*

Proof If f is invertible in $A(X)$ then it follows directly from the definition of $\mathfrak{Z}_A(f)$ that $\mathfrak{Z}_A(f) = \boldsymbol{Z}[X]$.

If f is not invertible in $A(X)$ then clearly $\varnothing \notin \mathfrak{Z}_A(f)$. Also, if $E \in \mathfrak{Z}_A(f)$ and $F \supset E$ then clearly $F \in \mathfrak{Z}_A(f)$. Now suppose $E_1, E_2 \in \mathfrak{Z}_A(f)$. We show that $E_1 \cap E_2 \in \mathfrak{Z}_A(f)$. To this end let H be a zero set with $H \subset (E_1 \cap E_2)^c$. We show that f is H-regular and it will follow that $E_1 \cap E_2 \in \mathfrak{Z}_A(f)$. Suppose, for the sake of contradiction, that f is not H-regular. It follows by Lemma 1.4 that $H \cap F \neq \varnothing$ for every $F \in \mathcal{Z}_A(f)$. So there is a z-filter $\mathcal{F}$ containing $\mathcal{Z}_A(f)$ and H, in particular $\mathcal{Z}_A(f) \subset \mathcal{F}$. By Lemma 1.1(b) it follows that $\lim_{\mathcal{F}} fh = 0$ for all $h \in A(X)$. Now note that the zero sets $H_1 = H \cap E_1$ and $H_2 = H \cap E_2$ are disjoint from E_2 and E_1, respectively, and so by hypothesis f is H_1-regular and H_2-regular. Since the collection of sets on which a given function is regular is closed under finite unions ([9], Lemma 1(b)), it follows that f is $H_1 \cup H_2$-regular. That is, there exists $k \in A(X)$ such that $fk(x) = 1$ for $x \in H_1 \cup H_2$. Since $H_1 \cup H_2 \subset H$ and since $\lim_{\mathcal{F}} fk = 0$ it follows that there is a zero set $H_0 \in \mathcal{F}$ with $H_0 \subset H - (H_1 \cup H_2)$ for which f is not H_0-regular. But since H_0 is a zero set which is disjoint from E_1 and E_2 it follows by the definition of $\mathfrak{Z}_A(f)$ that f is H_0-regular. This contradiction completes the proof. ■

Theorem 2.3 *Let $A(X)$ be a ring of continuous function such that $C^*(X) \subseteq A(X) \subseteq C(X)$. Then $A(X) = C(X)$ if and only if $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$ for all $f \in A(X)$.*

Proof Suppose $A(X) = C(X)$. Let $f \in A(X)$. If f is invertible in $A(X)$, then by Proposition 2.2 we have $\mathfrak{Z}_A(f) = \boldsymbol{Z}[X]$. Since f is invertible, $\boldsymbol{Z}(f)$ is empty, and hence $\langle \boldsymbol{Z}(f) \rangle = \boldsymbol{Z}[X]$. Now $f \in A(X) = C(X)$ is noninvertible if and only if $\boldsymbol{Z}(f) \neq \emptyset$. Suppose f is noninvertible. If H is a zero set such that $H \cap \boldsymbol{Z}(f) = \emptyset$, then f is H-regular in $C(X)$. So by the definition of $\mathfrak{Z}_A(f)$ it follows that $\boldsymbol{Z}(f) \in \mathfrak{Z}_A(f)$ and hence $\langle \boldsymbol{Z}(f) \rangle \subset \mathfrak{Z}_A(f)$. For the other containment, suppose $E \in \mathfrak{Z}_A(f)$ and $p \in E^c$. Since X is completely regular, there is a zero-set H containing p such that $H \subset E^c$. But f is invertible on H by definition of $\mathfrak{Z}_A(f)$, so $f \neq 0$ on H. It follows that f is nonzero on every point $p \in E^c$ and hence, $E \supseteq \boldsymbol{Z}(f)$. Thus, $E \in \langle \boldsymbol{Z}(f) \rangle$.

Conversely, suppose $f \in A(X)$ is never zero, that is $\boldsymbol{Z}(f) = \emptyset$. Then by hypothesis $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$ is not a z-filter, so f is invertible in $A(X)$ by Proposition 2.2. Thus $A(X)$ is *inverse closed* (every function in $A(X)$ that does not vanish on X has an inverse in $A(X)$). Any subring of $C(X)$ that is uniformly closed (closed in the uniform topology), contains all constant functions, and is inverse closed is $C(X)$ itself ([11], problem 44C part 5, p. 294). Since $A(X) \supseteq C^*(X)$, it is straight forward to see that $A(X)$ is uniformly closed. Since $A(X)$ also contains constant functions, it follows that $A(X) = C(X)$. ■

Corollary 2.4 *For any ideal $I \in C(X)$, $\mathfrak{Z}_C[I] = \boldsymbol{Z}[I]$.*

Proof It follows from Theorem 2.3 and the definitions that $\mathfrak{Z}_C[I] = \bigcup_{f \in I} \langle \boldsymbol{Z}(f) \rangle$. It remains to show that $\bigcup_{f \in I} \langle \boldsymbol{Z}(f) \rangle = \bigcup_{f \in I} \boldsymbol{Z}(f)$. The right to left containment follows directly from the definitions. The left to right containment also makes use of the fact that $\boldsymbol{Z}[I]$ is a z-filter and hence upward closed [6, p. 25]. ■

3. Comparing the correspondence for $C(X)$ and $C^*(X)$.

We pointed out in the introduction that the correspondences $\boldsymbol{Z}$ and $\boldsymbol{E}$, being defined on different rings, cannot be compared directly. In the next theorem, we compare these correspondences by comparing their extensions $\mathfrak{Z}_A$ and $\mathcal{Z}_A$ to intermediate rings. Indeed, we give a formula that relates $\mathfrak{Z}_A$ and $\mathcal{Z}_A$ for any ring $A(X)$ between $C^*(X)$ and $C(X)$. We need some notation.

For a z-filter $\mathcal{F}$ we write $h\mathcal{F}$ for the hull of $\mathcal{F}$, that is $h\mathcal{F}$ is the set of z-ultrafilters containing $\mathcal{F}$. If $\mathfrak{U}$ is a collection of z-ultrafilters we write $k\mathfrak{U}$ to denote the kernel of $\mathfrak{U}$, that is, $k\mathfrak{U}$ is the intersection of the z-ultrafilters in $\mathfrak{U}$.

Theorem 3.1 *Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$ and let f be a noninvertible function in $A(X)$. Then $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$.*

Proof Let $E \in \mathfrak{Z}_A(f)$. We show that $E \in \mathcal{U}$ for every z-ultrafilter $\mathcal{U}$ containing $\mathcal{Z}_A(f)$. Suppose that there exists $\mathcal{U} \in h\mathcal{Z}_A(f)$ such that $E \notin \mathcal{U}$. Then there exists $F \in \mathcal{U}$ such that $E \cap F = \varnothing$. But then f is F-regular by the definition of $\mathfrak{Z}_A(f)$. It follows that there exists an $h \in A(x)$ such that $\lim_{\mathcal{U}} fh \neq 0$, which is a contradiction to Lemma 1.1(b). For the other containment, suppose that $E \notin \mathfrak{Z}_A(f)$. Then there exists a zero-set $H \subset E^c$ such that f is not H-regular. By Lemma 1.4 it follows that H meets every $F \in \mathcal{Z}_A(f)$, and so there is a z-ultrafilter $\mathcal{U}$ containing H and $\mathcal{Z}_A(f)$. But then $E \notin \mathcal{U}$, and consequently $E \notin kh\mathcal{Z}_A(f)$. ■

Note that this result also gives an alternate proof of Proposition 2.2: if f is not invertible then $\mathcal{Z}_A(f)$ is a z-filter by Lemma 1.1(a). So $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$ is also a z-filter.

4. Ideals and $\mathfrak{Z}_A$.

The main goal of this section is to clarify the behavior of the correspondences $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ on maximal ideals in $A(X)$. It is known that $\boldsymbol{E}$ (resp. $\boldsymbol{Z}$) is a one-one correspondence between maximal ideals in $C^*(X)$ (resp. $C(X)$) and z-filters on X; it is noted in [6, p. 82] that it is remarkable that the characterization of the maximal ideals in $C^*(X)$ and in $C(X)$ have a common solution, that is, each maximal ideal in its ring corresponding to a z-ultrafilter on X. We mentioned earlier that for any $A(X)$, the map $\mathcal{Z}_A$ maps ideals in $A(X)$ to z-filters on X. It was furthermore shown in [2] that $\mathcal{Z}_A$ maps each maximal ideal in $A(X)$ to a z-filter that is contained in a unique z-ultrafilter on X. In [9] an inverse map $\mathcal{Z}_A^{\leftarrow}$ is defined, which in [10] is shown to map z-filters on X to ideals in $A(X)$. In this section, we show that the correspondence $\mathfrak{Z}_A$ indeed maps ideals in $A(X)$ to z-filters on X. We also show that it maps each maximal ideal to a z-filter contained in a unique z-ultrafilter, and that this containment can be strict for rings other than $C(X)$. Furthermore, we define an inverse map $\mathfrak{Z}_A^{\leftarrow}$ from z-filters to ideals. We show that when restricted to z-ultrafilters, $\mathfrak{Z}_A^{\leftarrow}$ coincides with the analogous inverse map $\mathcal{Z}_A^{\leftarrow}$, and that $\mathfrak{Z}_A^{\leftarrow}$ is a one-one correspondence between z-ultrafilters and maximal ideals.

We need some lemmas including some basic facts about the kernel-hull operator.

Lemma 4.1 *If $\mathcal{H}$ is a z-ultrafilter then for all z-filters $\mathcal{F}$ and $\mathcal{G}$, if $\mathcal{F} \wedge \mathcal{G} \subseteq \mathcal{H}$, then $\mathcal{F} \subseteq \mathcal{H}$ or $\mathcal{G} \subseteq \mathcal{H}$.*

Proof Suppose $\mathcal{F}$ and $\mathcal{G}$ are z-filters such that $\mathcal{F} \wedge \mathcal{G} \subseteq \mathcal{H}$. If $\mathcal{F} \nsubseteq \mathcal{H}$, then there exists $F \in \mathcal{F}$ such that $F \notin \mathcal{H}$. For all $G \in \mathcal{G}$, we have $F \cup G \in \mathcal{F}$ and $F \cup G \in \mathcal{G}$. Thus $F \cup G \in \mathcal{F} \wedge \mathcal{G} \subseteq \mathcal{H}$. Since $\mathcal{H}$ is a z-ultrafilter, z-ultrafilters are prime z-filters, and $F \notin \mathcal{H}$, it follows that $G \in \mathcal{H}$. Thus $\mathcal{G} \subseteq \mathcal{H}$. ■

Lemma 4.2 *Let $\mathcal{F}$ and $\mathcal{G}$ be z-filters on X. Then*

(a) $kh(\mathcal{F} \wedge \mathcal{G}) = kh\mathcal{F} \wedge kh\mathcal{G}$

(b) $kh(\mathcal{F} \vee \mathcal{G}) \supseteq kh\mathcal{F} \vee kh\mathcal{G}$

(c) $kh(\mathcal{F} \vee \mathcal{G}) = kh(kh\mathcal{F} \vee kh\mathcal{G})$

Proof (a) If $E \in kh(\mathcal{F} \wedge \mathcal{G})$ then E belongs to every z-ultrafilter that contains $\mathcal{F} \wedge \mathcal{G}$, so clearly E belongs to every z-ultrafilter that contains $\mathcal{F}$ and to every z-ultrafilter that contains $\mathcal{G}$; that is, $E \in kh\mathcal{F} \wedge kh\mathcal{G}$. For the other containment, suppose $E \in kh\mathcal{F} \wedge kh\mathcal{G}$. Then E belongs to every z-ultrafilter that contains $\mathcal{F}$ and to every z-ultrafilter that contains $\mathcal{G}$. Now let $\mathcal{U}$ be any z-ultrafilter that contains $\mathcal{F} \wedge \mathcal{G}$. Then by Lemma 4.1, either $\mathcal{F} \subseteq \mathcal{U}$ or $\mathcal{G} \subseteq \mathcal{U}$. In either case, $E \in \mathcal{U}$. Thus $E \in kh(\mathcal{F} \wedge \mathcal{G})$.

(b) Since $\mathcal{F} \subseteq \mathcal{F} \vee \mathcal{G}$ it follows that $kh\mathcal{F} \subseteq kh(\mathcal{F} \vee \mathcal{G})$. Similarly, $kh\mathcal{G} \subseteq kh(\mathcal{F} \vee \mathcal{G})$, and the result follows.

(c) By part (b), $kh(\mathcal{F} \vee \mathcal{G}) \supseteq kh\mathcal{F} \vee kh\mathcal{G} \supseteq \mathcal{F} \vee \mathcal{G}$. The result follows by taking kh of all three expressions and noting that kh is an idempotent operation. ■

Theorem 4.3 *Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$. If I is an ideal in $A(X)$, then $\mathfrak{Z}_A[I]$ is a z-filter on X. Moreover, if M is a maximal ideal in $A(X)$, then $\mathfrak{Z}_A[M]$ is contained in a unique z-ultrafilter on X.*

Proof Clearly $\emptyset \notin \mathfrak{Z}_A[I]$. Suppose $E \in \mathfrak{Z}_A[I]$ and $F \supseteq E$. Then $E \in \mathfrak{Z}_A(f)$ for some $f \in I$, so $F \in \mathfrak{Z}_A(f)$ and hence $F \in \mathfrak{Z}_A[I]$. If $E, F \in \mathfrak{Z}_A[I]$ then $E \in \mathfrak{Z}_A(f)$ and $F \in \mathfrak{Z}_A(g)$ for some $f, g \in I$. Now, using Theorem 3.1 and Lemma 4.2(b) we have $E \cap F \in \mathfrak{Z}_A(f) \vee \mathfrak{Z}_A(g) \subseteq kh(\mathcal{Z}_A(f) \vee \mathcal{Z}_A(g))$. But by Lemma 1.5 (a) (applied to f^2 and g^2) and Lemma 1.5(c) we have $kh(\mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)) = kh(\mathcal{Z}_A(f^2) \vee \mathcal{Z}_A(g^2)) = kh(\mathcal{Z}_A(f^2 + g^2)) = \mathfrak{Z}_A(f^2 + g^2) \subseteq \mathfrak{Z}_A[I]$. Thus $E \cap F \in \mathfrak{Z}_A[I]$. This shows that $\mathfrak{Z}_A[I]$ is an ideal.

Now, if M is a maximal ideal then $\mathcal{Z}_A[M]$ is contained in a unique z-ultrafilter $\mathcal{U}$ [2]. Since $\mathcal{Z}_A[M] \subseteq \mathfrak{Z}_A[M]$, it follows that $\mathfrak{Z}_A[M]$ is also contained in the z-ultrafilter $\mathcal{U}$. ■

The containment in Theorem 4.3 may be proper as the following example shows. For the example we use the fact that if $A(X) = C^*(X)$ then $\lim_{\mathcal{F}} fh = 0$ if and only if $\lim_{\mathcal{F}} f = 0$ for any $h \in C^*(X)$, and hence by Lemma 1.1(b), $\mathcal{Z}_A(f) \subseteq \mathcal{F}$ if and only if $\lim_{\mathcal{F}} f = 0$. We also make use of the following proposition, which is a slightly weaker form of Theorem 2.4 in [10]. It uses the inverse map of the set map $\mathcal{Z}_A$ defined by

$$\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] = \{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$$

where $\mathcal{F}$ is a z-filter on X.

PROPOSITION 4.4 *For any z-ultrafilter $\mathcal{U}$ and any ring of continuous functions $A(X)$, such that $C^*(X) \subseteq A(X) \subseteq C(X)$, the set $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal in $A(X)$.*

EXAMPLE 4.5 Let $A(X) = C^*[0,\infty)$. Let $E = \{1,2,3,\ldots\}$ and let $\mathcal{U}_E$ be any free z-ultrafilter on $[0,\infty)$ containing E. Let $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_E]$. By Proposition 4.4, M is a maximal ideal in $A(X)$. Now, by Theorem 4.3, $\mathfrak{Z}_A[M]$ is contained in the unique z-ultrafilter $\mathcal{U}_E$. We show that the containment is proper. In particular, we show that $E \notin \mathfrak{Z}_A[M]$. Now for each $f \in M$, by Lemma 1.1(b) and the above remarks, we have $\lim_{\mathcal{U}_E} f = 0$, in particular for each n, there is a set $U_n \in \mathcal{U}_E$ such that $-1/n < f(x) < 1/n$ for all $x \in U_n$. Select $a_1 \in U_1 \cap E$, and using the fact that $\mathcal{U}_E$ is free, select $B_1 \in \mathcal{U}_E$, such that $a_1 \notin B_1$. For each n, select $a_n \in U_n \cap E \cap \bigcap_{1\le j<n} B_j$ and select B_n such that $a_n \notin B_n$. Then (a_n) is a one-one E-valued sequence; in particular (a_n) assumes infinitely many integer values, and hence is unbounded. Furthermore, $f(a_n) \to 0$ as $n \to \infty$ by construction. Since f is continuous, we can choose distinct real values $b_n \notin E$ close to a_n (say, with $|a_n - b_n| < 1$), and such that $f(b_n) \to 0$ as $n \to \infty$. As the differences between the b_n and a_n are bounded, and the set $\{a_1, a_2, \ldots\}$ is unbounded, the set $F = \{b_1, b_2, \ldots\}$ is unbounded, and hence F is contained in some free z-ultrafilter $\mathcal{U}_F$. Then $\lim_{\mathcal{U}_F} f = 0$, and hence $\mathcal{Z}_A(f) \subseteq \mathcal{U}_F$ by Lemma 1.1(b). Since $E \notin \mathcal{U}_F$, $E \notin kh\mathcal{Z}_A(f) = \mathfrak{Z}_A(f)$. Since f was an arbitrary element of M it follows that $E \notin \mathfrak{Z}_A[M]$.

We now consider the inverse of the map $\mathfrak{Z}_A$.

DEFINITION 4.6 The inverse map of the set map $\mathfrak{Z}_A$ is defined by

$$\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}] = \{f \in A(X) : \mathfrak{Z}_A(f) \subseteq \mathcal{F}\}$$

where $\mathcal{F}$ is a z-filter on X.

THEOREM 4.7 *Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$. If $\mathcal{U}$ is a z-ultrafilter on X, then $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}] = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$. In particular, $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal in $A(X)$.*

PROOF We first observe that, in general, $\mathcal{Z}_A(f) \subseteq \mathcal{U}$ if and only if $\mathfrak{Z}_A(f) \subseteq \mathcal{U}$. Indeed, if $\mathcal{Z}_A(f) \subseteq \mathcal{U}$, then $\mathcal{U} \in h\mathcal{Z}_A(f)$, and by Theorem 3.1, it follows that $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f) \subseteq \mathcal{U}$. The converse is trivially true. From this we conclude that $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}] = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$. By Proposition 4.4, $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal, and this completes the proof. ■

THEOREM 4.8 *Let $A(X)$ be a ring of continuous functions such that $C^*(X) \subseteq A(X) \subseteq C(X)$. There is a one-one correspondence between z-ultrafilters on X and maximal ideals in $A(X)$ given by*

$$\mathcal{U} \to \mathfrak{Z}_A^{\leftarrow}[\mathcal{U}].$$

PROOF By Theorem 4.7, $\mathfrak{Z}_A^{\leftarrow}(\mathcal{U})$ is guaranteed to be a maximal ideal in $A(X)$. Thus it remains to show that $\mathfrak{Z}_A$ is one-one on the collection of z-ultrafilters on X. Now let $\mathcal{U}_1$ and $\mathcal{U}_2$ be z-ultrafilters on X, and suppose that $\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}_1] = \mathfrak{Z}_A^{\leftarrow}[\mathcal{U}_2]$. By Theorem 4.7, these are maximal ideals, so by Theorem 4.3, $\mathfrak{Z}_A\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}_1]$ is contained in a unique z-ultrafilter. In general, for any z-filter $\mathcal{F}$, we have directly from the definitions that $\mathfrak{Z}_A[\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}]] \subseteq \mathcal{F}$, and hence $\mathfrak{Z}_A\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}_1] \subseteq \mathcal{U}_1$. Similarly, $\mathfrak{Z}_A\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}_2]$ is contained in the unique z-ultrafilter $\mathcal{U}_2$. Since $\mathfrak{Z}_A\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}_1] = \mathfrak{Z}_A\mathfrak{Z}_A^{\leftarrow}[\mathcal{U}_2]$, it follows that $\mathcal{U}_1 = \mathcal{U}_2$. ■

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