
Sigma Algebras in Probabilistic Epistemic Dynamics

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Abstract

This paper extends probabilistic dynamic epistemic logic from a finite setting to an infinite setting, by introducing σ -algebras to the probability spaces in the models. This may extend the applicability of the logic to a real world setting with infinitely many possible measurements. It is shown that the dynamics preserves desirable properties of measurability and that completeness of the proof system holds with the extended semantics.

1 Introduction

A logic that combines probability, epistemics, and dynamics provides us with a tool for describing how qualitative and quantitative uncertainty change when given new information. A family of existing such logics [5, 9, 10], called *probabilistic dynamic epistemic logics*, stems from the paper [3] that introduced probabilistic epistemic logic, which can express interaction between qualitative and quantitative uncertainty. The setting of probabilistic epistemic logic in [3] involved probabilities over σ -algebras and potentially infinite sample spaces; however, the dynamic extensions in [5, 10] are restricted to a finite setting. An infinite setting allows us to model situations in which there may be measurements over infinitely many possibilities, and in which there may be, for example, a normal or uniform distribution over a continuum of possibilities. The paper [9] explored semantics and proof systems that extend the probabilistic dynamic epistemic logic in [5] to an infinite setting. Yet there remained questions, such as whether one can define the semantics in such a way that the dynamics preserves the property that the denotation of a formula can be assigned a probability (it was shown that such a property need not be preserved in the general semantics given in [9]), and whether the completeness proof of [5] can be adapted to the infi-

nite setting. In this paper, we answer these questions in the affirmative, but for an extension of the richer and more recently developed logic in [10], rather than for the extension of the logic in [5].

The logic in [10] involves three types of probabilities: prior, observation, and occurrence. *Prior probabilities* are subjective probabilities that agents had before an event. *Observation probabilities* are subjective probabilities that agents had about what event actually occurred. *Occurrence probabilities* are objective probabilities concerned with what event actually occurs. The prior probabilities lie in the original model, called a probabilistic epistemic model (the semantic structure in [3]), and the other two probabilities are given in another structure we call the *update frame*, which is combined with the prior model to form a new probabilistic epistemic model with posterior probabilities. The work in [9] did include prior and observation probabilities, but did not include occurrence probabilities, and this paper offers a framework that describes the interplay of all three types of probabilities in a setting in which the probabilities in the probabilistic epistemic models (the prior probabilities) are potentially infinite.

We present in the next section probabilistic epistemic logic, which will be the foundation for the dynamic extension this paper will focus on. In addition, the semantic structures, probabilistic epistemic models, will be the structures comprising the semantics that all the languages in this paper will be interpreted over. These structures will involve probabilities defined on sigma algebras, and will be in an infinite setting. Section 3 will introduce our version of dynamic probabilistic epistemic logic, by first introducing update frames and then by providing the semantics. It is shown in this section (Theorem 3.4) that dynamics preserves the property that the denotation of every formula is measurable. Section 4 will introduce a sound and complete proof system for the logic with respect to the semantics given in Section 3.

2 Probabilistic Epistemic Logic

The logic we will focus on, which we call *Update Logic*, builds on Probabilistic Epistemic Logic, which we cover in this section.

We first provide some definitions.

- A σ -algebra over a set S is a non-empty set of subsets of S closed under countable unions and complements.
- Given a σ -algebra \mathcal{A} , a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ is called a *measure* if $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for pairwise disjoint $A_i \in \mathcal{A}$. This condition is called countable additivity.
- A *finite measure* is a measure such that all of its values are finite.
- A *measure space* is a tuple (S, \mathcal{A}, μ) , where S is a set, \mathcal{A} is a σ -algebra over S , and $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a measure.

We focus throughout this paper on a variation of probability spaces that also includes the 0 function. Given a set X , let $\text{spaces}(X)$ be the set of measure spaces (S, \mathcal{A}, μ) , such that $S \subseteq X$ and μ is a finite measure normalized to 0 or 1, that is, $\mu(S)$ is either 0 or 1.

The structures over which we will interpret formulas have *Kripke frames* as a core component.

Definition 2.1 (Kripke frame). Given a set \mathbb{I} , which we will associate with a group of agents, a *Kripke frame for \mathbb{I}* is a tuple $F = (X, R)$, where

- X is a set of *states* called the *carrier set* of F , and
- $R : \mathbb{I} \rightarrow X \times X$ is a function assigning a binary relation R_ℓ to each agent in $\ell \in \mathbb{I}$.

We often write the superscript F for each component to emphasize which structure the components are in. That is, we might write X^F for X and R^F for R . Since we are in a multi-agent setting, we call the relations R_ℓ *epistemic relations*, and view $xR_\ell y$ to mean that ℓ considers y possible given x . But while epistemic relations are generally viewed as equivalence relations, we for simplicity do not impose any restrictions on them.

When we add to a Kripke frame a component that decorates each state with properties called *atomic propositions* and another component that decorates states with measure spaces, we obtain the type of structure over which we will interpret our formulas. We call these structures *probabilistic epistemic models*.

Definition 2.2 (Probabilistic Epistemic Model). Given a set AtProp of *atomic propositions* and a set

\mathbb{I} of *agents*, we define a *probabilistic epistemic model for AtProp and \mathbb{I}* as a tuple $M = (X, R, V, \mathbb{P})$, where

- (X, R) is a Kripke frame for \mathbb{I} ,
- $V : \text{AtProp} \rightarrow \mathcal{P}(X)$ is a “valuation” function, and
- $\mathbb{P} : \mathbb{I} \times X \rightarrow \text{spaces}(X)$ is a function associating every agent ℓ and element x with a measure space $\mathbb{P}_{\ell,x} = (S_{\ell,x}, \mathcal{A}_{\ell,x}, \mu_{\ell,x})$, where $S_{\ell,x} \subseteq X$.

A pair (M, x) with $x \in X$ is called a *pointed model*.

The measure space $\mathbb{P}_{\ell,x}$ represents agent ℓ ’s subjective view of the likelihoods of certain subsets of the model if the actual state were x . We generally omit the superscripts M when they are understood from context. Similarly we may omit explicit mention of AtProp and \mathbb{I} when they are understood from context. We may also omit the σ -algebra component of the measure space $\mathbb{P}_{\ell,x}$, when that component is just the same as the power set of the sample space. A Kripke frame or a probabilistic epistemic model is said to be *finite* if its carrier set is finite.

2.1 Probabilistic Epistemic Language

Definition 2.3 (Language \mathcal{L}_{PEL}). Given a set \mathbb{I} of agents and a set AtProp of atomic propositions, the language $\mathcal{L}_{\text{PEL}}(\mathbb{I}, \text{AtProp})$ is given by the following Backus-Naur form:

$$\varphi ::= \text{true} \mid \text{false} \mid p \mid \neg\varphi \mid \varphi \star \varphi \mid [\ell]\varphi \mid t_\ell \geq q$$

where $p \in \text{AtProp}$, $\ell \in \mathbb{I}$, $q \in \mathbb{Q}$, and $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$; for each agent $\ell \in \mathbb{I}$,

$$t_\ell ::= qP_\ell(\varphi) \mid t_\ell + t_\ell$$

where $q \in \mathbb{Q}$. We simply write \mathcal{L}_{PEL} for $\mathcal{L}_{\text{PEL}}(\mathbb{I}, \text{AtProp})$ when \mathbb{I} and AtProp are clear from the context or are immaterial.

The formula **false** is not true anywhere, while **true** is true everywhere. The operators \neg , \wedge , \vee , \rightarrow , and \leftrightarrow have their usual boolean readings of *not*, *and*, *or*, *only if*, and *if and only if* respectively. The formula $[\ell]\varphi$ is read as *agent ℓ believes φ* . The formula $q_1P_\ell(\varphi_1) + \dots + q_nP_\ell(\varphi_n) \geq q$ states that a linear combination of probabilities for a specific agent is at least q . We use linear combinations for technical reasons. In the linear combination, $P_\ell(\varphi_i)$ stands for the probability that agent ℓ assigns to the truth of formula φ_i . We involve addition of probability terms, since additivity is a fundamental condition on measures and hence on probabilities. We restrict the probability terms of a particular probability formula to ones that involve a

single agent for simplicity. The formulas inside the arguments of the probability terms may involve other probability formulas for other agents.

Abbreviations: We have the usual modal abbreviation: $\langle \ell \rangle \varphi \stackrel{\text{def}}{=} \neg[\ell]\neg\varphi$ (read as *agent ℓ considers φ possible*). Also, if $c \in \mathbb{Q}$ and $t = \sum_{k=1}^n q_k \mathbf{P}_\ell(\varphi_k) = q_1 \mathbf{P}_\ell(\varphi_1) + \dots + q_n \mathbf{P}_\ell(\varphi_n)$, we have the following abbreviations.

$$\begin{aligned} ct &\stackrel{\text{def}}{=} \sum_{k=1}^n cq_k \mathbf{P}_\ell(\varphi_k) \\ t \leq q &\stackrel{\text{def}}{=} -t \geq -q \\ t < q &\stackrel{\text{def}}{=} \neg(t \geq q) \\ t > q &\stackrel{\text{def}}{=} \neg(t \leq q) \\ t = q &\stackrel{\text{def}}{=} t \leq q \wedge t \geq q \\ t \geq s &\stackrel{\text{def}}{=} t - s \geq 0 \\ t = s &\stackrel{\text{def}}{=} t - s \geq 0 \wedge s - t \geq 0 \end{aligned}$$

Semantics

We will define $\llbracket \cdot \rrbracket_{\mathcal{L}_{\text{PEL}}}$ as a function mapping \mathcal{L}_{PEL} to functions that map probabilistic epistemic models to subsets of their carrier set. We write $\llbracket \varphi \rrbracket^M$ for $\llbracket \varphi \rrbracket(M)$, and the set denoted $\llbracket \varphi \rrbracket^M$ is considered to consist of all states in M at which φ is true. Our probability formulas will involve applying measures $\mu_{\ell,x}^M$ to sets of the form $\llbracket \varphi \rrbracket^M$, as long as $\llbracket \varphi \rrbracket^M \cap S_{\ell,x}^M \in \mathcal{A}_{\ell,x}^M$. Two options for addressing the requirement that $\llbracket \varphi \rrbracket^M \cap S_{\ell,x}^M \in \mathcal{A}_{\ell,x}^M$ are as follows. One is to place a restriction on the models such that $\llbracket \varphi \rrbracket^M \cap S_{\ell,x}^M \in \mathcal{A}_{\ell,x}^M$ holds for every φ , ℓ and x . Another is to involve a commonly used extension of the notion of a measure, called the outer measure, which extends the domain of the measure to the *whole power set* of the sample space (rather than just the sets in the σ -algebra). Given a measure $\mu : \mathcal{A} \rightarrow [0, \infty)$, the *outer measure extension* of μ is defined to be $\mu^* : \mathcal{P}(S) \rightarrow [0, \infty)$, given by

$$\mu^*(A) = \inf\{\mu(B) \mid B \in \mathcal{A}, A \subseteq B\}. \quad (1)$$

The outer measure extension is indeed an extension [4]: $\mu(A) = \mu^*(A)$ for each $A \in \mathcal{A}$. We define the semantics in the more general setting using the outer measure, and since the outer measure extends the measure, the semantics behaves the same way on the restricted input model if we replace the outer measure with the original measure.

We define $\llbracket \cdot \rrbracket_{\mathcal{L}_{\text{PEL}}}$ using the following rules:

$\llbracket \text{true} \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$	$= X^M$
$\llbracket p \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$	$= V^M(p)$
$\llbracket \neg\varphi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$	$= X^M - \llbracket \varphi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$
$\llbracket \varphi \wedge \psi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$	$= \llbracket \varphi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M \cap \llbracket \psi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$
$\llbracket \varphi \vee \psi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$	$= \llbracket \varphi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M \cup \llbracket \psi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$
$\llbracket \varphi \rightarrow \psi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$	$= \llbracket \neg\varphi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M \cup \llbracket \psi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$
$\llbracket \varphi \leftrightarrow \psi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$	$= (\llbracket \varphi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M \cap \llbracket \psi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M) \cup (\llbracket \neg\varphi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M \cap \llbracket \neg\psi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M)$
$\llbracket [\ell]\varphi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$	$= \{x \mid R_\ell^M(x) \subseteq \llbracket \varphi \rrbracket_{\mathcal{L}_{\text{PEL}}}^M\}$
$\llbracket q_1 \mathbf{P}_\ell(\varphi_1) + \dots + q_n \mathbf{P}_\ell(\varphi_n) \geq q \rrbracket_{\mathcal{L}_{\text{PEL}}}^M$	$= \{x \mid \sum_{k=1}^n q_k (\mu_{\ell,x}^M)^*(\llbracket \varphi_k \rrbracket_{\mathcal{L}_{\text{PEL}}}^M \cap S_{\ell,x}^M) \geq q\}$

We often omit the subscript \mathcal{L}_{PEL} and/or the superscript M if they should be understood from context.

2.2 Comparing probability and epistemics

It might seem natural to define belief by agent ℓ in the truth of a formula as that formula having probability 1 for agent ℓ . This might be difficult to enforce when the probability measure is, for example, the uniform measure on the unit interval. For the uniform measure on the unit interval, what would be the set of related states? Surely it should have probability 1, but given any set A of probability 1, one can remove countably many states and still have a set A' with probability 1. If we make the set of possibilities to be the set A , then a formula true and only true on A' would have probability 1, but would not be believed. But the situation improves if the probability measure has a countable σ -algebra and if the set of states at which a formula is true is in the σ -algebra. Then we could define the set of related states to be those states not contained in a set of measure 0. Note that this set cannot be empty, since if every element were contained in a set of measure 0, then the subadditivity condition of measures would ensure that the sample space also have measure 0, which would contradict the fact we are assuming the measure is normalized to 1. By defining the set of related states to be those states not contained in any set of measure 0, we obtain the following two relationships:

- $\langle \ell \rangle \varphi \leftrightarrow \mathbf{P}_\ell(\varphi) > 0$ and
- $[\ell]\varphi \leftrightarrow \mathbf{P}_\ell(\varphi) = 1$.

As pointed out in [3], however, it might be helpful to use the flexibility of the two components, and not to impose the restriction that information believed is identified with information given probability 1. The example presented in the above-cited reference involves an event, namely a bit being output by a computer, and asserts that that this event should not be

assigned a probability, since too little is known about how the computer determines what bit to output. In addition, the example considers the flipping of a fair coin, where the odds are known to be even for heads and tails. Although the probability of the bit is unknown, it can be determined that the probability of the combination of the bit being 1 and the coin flip resulting in heads is 0.5. In order to have a model that assigns this combination a probability but to still reflect uncertainty about the coin flip, the example considers a model in which the agent is epistemically uncertain about two probability spaces: one space has as its sample space outcomes where the bit is always 1, and the other space has as its sample space outcomes where the bit is always 0. Thus probabilistic epistemic logic allows us to model agents who are subjectively uncertain about probability spaces.

3 Update Logic

In this section, we adapt the probabilistic dynamic epistemic logic from [10] to an infinite setting. Probabilistic dynamic epistemic logic adds to probabilistic epistemic logic operators of the form $[U, e]\varphi$, where (U, e) is a structure called a pointed update frame. A pointed update frame represents a communication that transforms one probabilistic epistemic model into another. We use essentially the same update frames as in [10]; where we differ is that we, unlike in [10], will use the probabilistic epistemic models of the previous section, that use σ -algebras.

Definition 3.1 (\mathcal{L} -update frame). Given a set \mathbb{I} of agents and a language \mathcal{L} that extends propositional logic¹, an \mathcal{L} -update frame for \mathbb{I} is a tuple $U = (X, R, \Phi, \mathbb{P})$ such that

- (X, R) is a finite Kripke frame,
- Φ is a finite² set of pairwise inconsistent³ formulas called *preconditions*,
- $\mathbb{P} : (\mathbb{I} \times X) \cup \Phi \rightarrow \text{spaces}(X)$ is a function associating every agent ℓ and element x with a probability measure space $\mathbb{P}_{\ell, e} = (X, \mathcal{P}(X), \mu_{\ell, e})$ and every precondition $\varphi \in \Phi$ with a probability measure space $\mathbb{P}_{\varphi} = (X, \mathcal{P}(X), \mu_{\varphi})$.

A pair (U, e) with $e \in X$ is called a *pointed update frame*, with e being called the point of the pair (U, e) .

¹We really intend for the language to extend \mathcal{L}_{PEL} , but we emphasize propositional logic, since we will make use of disjunctions of \mathcal{L} formulas in forming notation used throughout this section.

²The restriction that this set be finite is closely related to the fact that we are working with finitary languages.

³Pairwise inconsistent formulas guarantee that their denotations be pairwise disjoint.

We often write $\mu_{\varphi}(e)$ for $\mu_{\varphi}(\{e\})$, and similarly for other measures defined on singletons. The probabilities $\mathbb{P}_{\ell, e}$ are called observation probabilities, and \mathbb{P}_{φ} are called occurrence probabilities. Occurrence probabilities represent the likelihood of the event actually happening. An example is given in [10], where a person wants to determine whether he has a disease by observing a likely symptom of the disease, the symptom being the swelling of a gland. We consider two preconditions: having the disease and not having the disease. Hence we have two occurrence probability distributions. One would assume the precondition that the person has the disease and would give the likelihood that he has the symptom, most likely large but with a probability less than 1. The other would assume the precondition that he does not have the disease and would give the likelihood that he has the symptom, most likely small, but with a probability greater than 0.

We define a function $\text{pre} : X \rightarrow \mathcal{L}$, given by $\text{pre}(e) = \bigvee \{\varphi \mid \mu_{\varphi}(e) > 0\}$, where the empty disjunction is the formula **false**, which is not satisfied by any state in any model. Again, we often add superscripts of U to clarify which structure components are in.

In what follows, when given a probabilistic epistemic model M , for each $s \in \bigcup_{\varphi \in \Phi} \llbracket \varphi \rrbracket^M$, let $\varphi(s)$ be the precondition $\varphi \in \Phi$ for which $s \in \llbracket \varphi \rrbracket^M$. Also, for each set $A \subseteq X^M \times X^U$, let $A_e = \{(x, f) \in A \mid f = e\}$, and let π_j be the projection function taking an tuple and returning its j^{th} component.

Definition 3.2 (Update product). The *update product* between a probabilistic epistemic model M and an \mathcal{L} -update frame U is the probabilistic epistemic model $M[U]$ with the first three components:

- $X^{M[U]} = \{(x, e) \mid x \in \llbracket \text{pre}(e) \rrbracket^M\}$,
- $(x, e)R_{\ell}^{M[U]}(y, f)$ iff $xR_{\ell}^M y$ and $eR_{\ell}^U f$,
- $\llbracket p \rrbracket^{M[U]} = \{(x, e) \in X^{M[U]} \mid x \in \llbracket p \rrbracket^M\}$,

and the last component

$$\mathbb{P}_{\ell, (x, e)}^{M[U]} = (S_{\ell, (x, e)}^{M[U]}, \mathcal{A}_{\ell, (x, e)}^{M[U]}, \mu_{\ell, (x, e)}^{M[U]})$$

defined by:

- $S_{\ell, (x, e)}^{M[U]} = \{(y, e) \in X^{M[U]} \mid y \in S_{\ell, x}^M\}$,
- $\mathcal{A}_{\ell, (x, e)}^{M[U]}$ is the smallest σ -algebra containing

$$\{A \times B \mid A \in \mathcal{A}_{\ell, x}^M, B \in \mathcal{P}(X^U)\} \cap \mathcal{P}((X^{M[U]})^2),$$

- and

$$\mu_{\ell, (x, e)}^{M[U]}(C) = N/D, \quad (2)$$

where

$$N = \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_{\varphi}^U(\{f\}) \cdot \mu_{\ell,e}^U(\{f\}) \cdot \mu_{\ell,x}^M(\llbracket \varphi \wedge \text{pre}(f) \rrbracket^M \cap S_{\ell,x}^M \cap \pi_1(C))$$

and

$$D = \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_{\varphi}^U(\{f\}) \cdot \mu_{\ell,e}^U(\{f\}) \cdot \mu_{\ell,x}^M(\llbracket \varphi \wedge \text{pre}(f) \rrbracket^M \cap S_{\ell,x}^M),$$

if the denominator D is not 0; otherwise $\mu_{\ell,(x,e)}^{M[U]} = 0$, the zero measure.

The function $\mu_{\ell,(x,e)}^{M[U]}$ is a measure, since it is a finite linear combination of measures $\mu_{\ell,x}$ restricted to the domains of the form $\llbracket \varphi \wedge \text{pre}(f) \rrbracket^M \cap S_{\ell,x}$. The denominator is the same as the numerator except that it is given the largest input under this domain restriction, and hence represents the largest value the numerator can obtain. If the denominator is non-zero, this normalizes the quotient to a probability measure.

3.1 Update Language

The following definition is adapted from [6, 7].

Definition 3.3 (Language \mathcal{L}_{UL}). Given a set \mathbb{I} of agents and a set AtProp of atomic propositions, the language $\mathcal{L}_{\text{UL}}(\mathbb{I}, \text{AtProp})$ is given by adding to the Bachus-Naur form of \mathcal{L}_{PEL} the formula $[U, e]\varphi$, where (U, e) is an \mathcal{L}_{UL} -update frame.

Note that there are two sources of mutual recursion in this definition. One, which is inherited from probabilistic epistemic logic, is between terms and formulas. The other, which is inherited from dynamic epistemic logic (which has not been defined in this paper), is between formulas and update frames. The Bachus-Naur form generates the smallest sets of formulas, terms, and pointed update frames for which this mutual recursion holds.

The semantics of \mathcal{L}_{UL} is obtained by adding to the semantics of \mathcal{L}_{PEL} the following rule:

$$\llbracket [U, e]\varphi \rrbracket^M = \llbracket \neg \text{pre}(e) \rrbracket^M \cup \{x \mid (x, e) \in \llbracket \varphi \rrbracket^{M[U]}\}.$$

Note that $\llbracket \neg \text{pre}(e) \rrbracket^M \cup \{x \mid (x, e) \in \llbracket \varphi \rrbracket^{M[U]}\} = \llbracket \neg \text{pre}(e) \rrbracket^M \cup \pi_1(\llbracket \varphi \rrbracket_e^{M[U]})$. We may often write $\llbracket \varphi \rrbracket$ for $\llbracket \varphi \rrbracket^M$, when the model M is clear from context.

3.2 Preserving measurability through updating

Here we show that the dynamics of the semantics preserves the property that a formula always denotes a

measurable set. To be more precise, let mea be a function from probabilistic epistemic models M to $\mathcal{P}(\mathcal{P}(X^M))$, such that

$$\text{mea}(M) = \{Z \subseteq X^M \mid Z \cap S_{\ell,x}^M \in \mathcal{A}_{\ell,x}^M \ \forall x \in X^M, \ell \in \mathbb{I}\}.$$

Note that $\text{mea}(M)$ is a σ -algebra.

Theorem 3.4. *Suppose M is such that for every formula $\varphi \in \mathcal{L}_{\text{UL}}$, $\llbracket \varphi \rrbracket^M \in \text{mea}(M)$. Then for any update frame U and formula φ , we have that $\llbracket \varphi \rrbracket^{M[U]} \in \text{mea}(M[U])$.*

Proof Suppose M is such that for every formula $\varphi \in \mathcal{L}_{\text{UL}}$, $\llbracket \varphi \rrbracket^M \in \text{mea}(M)$. Fix an update frame U . Given a formula φ , we have that

$$\llbracket \varphi \rrbracket^{M[U]} = \bigcup_{e \in X^U} \llbracket \varphi \rrbracket_e^{M[U]}. \quad (3)$$

Now for each e , we have that

$$\llbracket [U, e]\varphi \rrbracket^M \stackrel{\text{def}}{=} \llbracket \neg \text{pre}(e) \rrbracket^M \cup \pi_1(\llbracket \varphi \rrbracket_e^{M[U]}) \in \text{mea}(M).$$

Thus, as $\llbracket \neg \text{pre}(e) \rrbracket^M \cap \pi_1(\llbracket \varphi \rrbracket_e^{M[U]}) = \emptyset$ and $\llbracket \neg \text{pre}(e) \rrbracket^M \in \text{mea}(M)$,

$$\pi_1(\llbracket \varphi \rrbracket_e^{M[U]}) \in \text{mea}(M).$$

Then using the definition of each $\mathcal{A}_{\ell,(x,e)}^{M[U]}$,

$$\pi_1(\llbracket \varphi \rrbracket_e^{M[U]}) \times \{e\} \in \text{mea}(M[U]).$$

As $\pi_1(\llbracket \varphi \rrbracket_e^{M[U]}) \times \{e\} = \llbracket \varphi \rrbracket_e^{M[U]}$ we have that $\llbracket \varphi \rrbracket_e^{M[U]} \in \text{mea}(M[U])$. As $\text{mea}(M[U])$ is a σ -algebra, and in particular is closed under finite unions, we have by (3) that $\llbracket \varphi \rrbracket^{M[U]} \in \text{mea}(M[U])$. \dashv

4 Proof Systems

We now present a proof system for the validity of \mathcal{L}_{UL} . The proof system, adapted from [9, 10], is given in Figure 1. The schemes CL through UK, as well as the rules MP, $[\ell]$ -NEC, and $[U, s]$ -NEC are standard in dynamic epistemic logic. The axiom UP describes the updating of a probability formula, and asserts that the update is equivalent to two disjuncts. The first disjunct asserts that if the probability of the denominator of (2) is 0, then the updated measure is the 0 function. The second disjunct asserts that if the denominator of (2) is positive, then the updated measure is given by (2). The axiom UP is essentially the same as that in [10], except it accounts for the differences in the update semantics that allow for σ -algebras to be involved in the input model. The schemes IP through P3 come from [3], with the exception of P2, which addresses the possibilities of our measures being the 0

AXIOM SCHEMES

CL.	Schemes for Classical Propositional Logic
KK	$[\ell](\varphi \rightarrow \psi) \rightarrow ([\ell]\varphi \rightarrow [\ell]\psi)$
KU	$[U, e](\varphi \rightarrow \psi) \rightarrow ([U, e]\varphi \rightarrow [U, e]\psi)$
UA	$[U, e]p \leftrightarrow (\text{pre}(e) \rightarrow p)$ (p an atomic formula)
UC	$[U, e](\varphi \wedge \psi) \leftrightarrow [U, e]\varphi \wedge [U, e]\psi$
UN	$[U, e]\neg\varphi \leftrightarrow (\text{pre}(e) \rightarrow \neg[U, e]\varphi)$
UK	$[U, e][\ell]\varphi \leftrightarrow (\text{pre}(e) \rightarrow \bigwedge_{f: eR_\ell f} [\ell][U, f]\varphi)$
UP	$[U, e](\sum_{k=1}^n q_k \mathbf{P}_\ell(\psi_k) \geq q) \leftrightarrow$ $t = 0 \wedge (\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \mathbf{P}_\ell(\text{false}) \geq q) \vee$ $\neg(t = 0) \wedge$ $(\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \sum_{\varphi \in \Phi, f \in X^U} \mu_\varphi(f) \mu_{\ell, e}(f) \mathbf{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f]\psi_k) \geq qt),$ where $t = \sum_{\varphi \in \Phi, f \in X^U} \mu_\varphi(f) \mu_{\ell, e}(f) \mathbf{P}_\ell(\varphi \wedge \text{pre}(f))$
IP	$\sum_{k=1}^n q_k \mathbf{P}_\ell(\varphi_k) \geq q \rightarrow \sum_{k=1}^n q_{j_k} \mathbf{P}_\ell(\varphi_{j_k}) \geq q$ for permutation j_1, \dots, j_n of $1, \dots, n$
I0	$t \geq q \leftrightarrow t + 0 \mathbf{P}_\ell(\varphi_{k+1}) \geq q$
IA	$\sum_{k=1}^n q_k \mathbf{P}_\ell(\varphi_k) \geq q \wedge \sum_{k=1}^n q'_k \mathbf{P}_\ell(\varphi_k) \geq q' \rightarrow$ $\sum_{k=1}^n (q_k + q'_k) \mathbf{P}_\ell(\varphi_k) \geq (q + q')$
IM	$t \geq q \leftrightarrow dt \geq dq$ where $d > 0$
DICH	$(t \geq q) \vee (t \leq q)$
MON	$(t \geq q) \rightarrow (t > q')$ where $q > q'$
P1	$\mathbf{P}_\ell(\varphi) \geq 0$
P2	$\mathbf{P}_\ell(\text{true}) = 1 \vee \mathbf{P}_\ell(\text{true}) = 0$
P3	$\mathbf{P}_\ell(\varphi \wedge \psi) + \mathbf{P}_\ell(\varphi \wedge \neg\psi) = \mathbf{P}_\ell(\varphi)$

RULES

$$\frac{\vdash \varphi \rightarrow \psi \quad \vdash \varphi}{\vdash \psi} \text{(MP)}$$

$$\frac{\vdash \varphi}{\vdash [\ell]\varphi} \text{([\ell]-NEC)}$$

$$\frac{\vdash \varphi}{\vdash [U, e]\varphi} \text{([U, e]-NEC)}$$

$$\frac{\vdash \varphi \leftrightarrow \psi}{\vdash \mathbf{P}_\ell(\varphi) = \mathbf{P}_\ell(\psi)} \text{(EQV)}$$

Figure 1: The Proof System

measure. This was suggested in [9], but completeness was not addressed in that reference. Finally the rule EQV comes from [5, 9], and plays the same role as the distributivity scheme in [3].

Theorem 4.1. *The proof system given in Figure 1 is sound for validity of \mathcal{L}_{UL} .*

Proof We limit ourselves to presenting the proofs for axioms P2 and UP, which we consider in turn below.

AXIOM P2 We prove the soundness of this axiom for an arbitrary probabilistic epistemic model M . To this end, we reason as follows:

$$\begin{aligned} &= \llbracket \mathbf{P}_\ell(\text{true}) = 1 \vee \mathbf{P}_\ell(\text{true}) = 0 \rrbracket^M \\ &= \llbracket \mathbf{P}_\ell(\text{true}) = 1 \rrbracket^M \cup \llbracket \mathbf{P}_\ell(\text{true}) = 0 \rrbracket^M \\ &= \{x \mid (\mu_{\ell, x}^M)^*(\llbracket \text{true} \rrbracket^M \cap S_{\ell, x}) = 1\} \\ &= \cup \{x \mid (\mu_{\ell, x}^M)^*(\llbracket \text{true} \rrbracket^M \cap S_{\ell, x}) = 0\} \\ &= \{x \mid (\mu_{\ell, x}^M)^*(S_{\ell, x}) = 1\} \cup \{x \mid (\mu_{\ell, x}^M)^*(S_{\ell, x}) = 0\} \\ &= \{x \mid \mu_{\ell, x}^M(S_{\ell, x}) = 1\} \cup \{x \mid \mu_{\ell, x}^M(S_{\ell, x}) = 0\} \\ &= \{x \mid \mu_{\ell, x}^M(S_{\ell, x}) = 1 \text{ or } \mu_{\ell, x}^M(S_{\ell, x}) = 0\} \\ &= X^M. \end{aligned}$$

AXIOM UP Let U be an \mathcal{L} -update frame for \mathbb{I} . We limit ourselves to proving that UP is valid over M when $n = 1$. The general case is only notationally more complex.

As in the statement of axiom UP, let

$$t = \sum_{\varphi \in \Phi, f \in X^U} \mu_\varphi(f) \mu_{\ell, e}(f) \mathbf{P}_\ell(\varphi \wedge \text{pre}(f))$$

We wish to show that the denotation on the left-hand side of the implication is the same as the denotation on the right-hand side of the implication. Let (M, x) be an arbitrary pointed model, and consider two cases depending on whether $x \in \llbracket t = 0 \rrbracket$.

Case $x \in \llbracket t = 0 \rrbracket$. We claim that

$$\begin{aligned} &\llbracket (t = 0) \wedge [U, e](q_1 \mathbf{P}_\ell(\psi_k) \geq q) \rrbracket^M \\ &= \llbracket (t = 0) \wedge (\text{pre}(e) \rightarrow q_1 \mathbf{P}_\ell(\text{false}) \geq q) \rrbracket^M. \end{aligned}$$

If $x \notin \text{pre}(e)$, then the right conjunct of each side of the equation is vacuously true, and hence the equation holds. Let us assume that $x \in \text{pre}(e)$. First note that $x \in \llbracket t = 0 \rrbracket^M$ if and only if

$$\begin{aligned} &\sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi^U(\{f\}) \mu_{\ell, e}^U(\{f\}) \\ &\mu_{\ell, x}^M(\llbracket \varphi \wedge \text{pre}(f) \rrbracket^M \cap S_{\ell, x}^M) = 0. \end{aligned}$$

This is immediate from the fact that the set $\llbracket t = 0 \rrbracket^M$ is equal to

$$\begin{aligned} &\{x \mid \sum_{(\varphi, f) \in \Phi \times X^U} \mu_\varphi^U(\{f\}) \mu_{\ell, e}^U(\{f\}) \\ &\quad (\mu_{\ell, x}^M)^*(\llbracket \varphi \rrbracket^M \cap \llbracket \text{pre}(f) \rrbracket^M \cap S_{\ell, x}^M) = 0\}. \end{aligned}$$

Then by Definition 3.2, $\mu_{\ell, (x, e)}^{M[U]}$ is the zero measure. Thus under the assumption that $x \in \llbracket \text{pre}(e) \rrbracket^M$, $x \in \llbracket (t = 0) \wedge [U, e](q_1 \text{P}_\ell(\psi_k) \geq q) \rrbracket^M$ if and only if $q \leq 0$. For the right hand side, it is easy to see that under the assumption that $x \in \llbracket \text{pre}(e) \rrbracket^M$, $x \in \llbracket (t = 0) \wedge (\text{pre}(e) \rightarrow q_1 \text{P}_\ell(\text{false}) \geq q) \rrbracket^M$ if and only if $q \leq 0$, since $\mu_{\ell, x}(\llbracket \text{false} \rrbracket) = \mu_{\ell, x}(\emptyset) = 0$.

Case $x \notin \llbracket t = 0 \rrbracket$. We claim that

$$\begin{aligned} & \llbracket \neg(t = 0) \wedge [U, e](q_1 \text{P}_\ell(\psi_k) \geq q) \rrbracket^M = \\ & \llbracket \neg(t = 0) \wedge (\text{pre}(e) \rightarrow \\ & q_1 \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi(f) \mu_{\ell, e}(f) \\ & \text{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f]\psi_k) \geq qt) \rrbracket^M. \end{aligned}$$

Like in the case where $x \in \llbracket t = 0 \rrbracket^M$, if $x \notin \llbracket \text{pre}(e) \rrbracket^M$, this equality is immediate. So assume that $x \in \llbracket \text{pre}(e) \rrbracket^M$ as well as $x \notin \llbracket t = 0 \rrbracket$. Then we aim to show that for such x , the following are equivalent:

1. $x \in \llbracket [U, e](q_1 \text{P}_\ell(\psi_k) \geq q) \rrbracket^M$
2. $x \in \llbracket q_1 \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi(f) \mu_{\ell, e}(f) \text{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f]\psi_k) \geq qt \rrbracket^M$

First note that $x \in \llbracket [U, e](q_1 \text{P}_\ell(\psi_k) \geq q) \rrbracket^M$ is equivalent to

$$x \in \pi_1(\llbracket q_1 \text{P}_\ell(\psi) \geq q \rrbracket_e^{M[U]}). \quad (4)$$

and by Definition 3.2, this is equivalent to $q_1 N/D \geq q$, where

$$\begin{aligned} N &= \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi^U(\{f\}) \mu_{\ell, e}^U(\{f\}) \\ & \mu_{\ell, x}^M(\llbracket \varphi \wedge \text{pre}(f) \rrbracket^M \cap S_{\ell, x}^M \cap \pi_1(\llbracket \psi \rrbracket^{M[U]})) \end{aligned}$$

and

$$\begin{aligned} D &= \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi^U(\{f\}) \mu_{\ell, e}^U(\{f\}) \\ & \mu_{\ell, x}^M(\llbracket \varphi \wedge \text{pre}(f) \rrbracket^M \cap S_{\ell, x}^M). \end{aligned}$$

Multiplying both sides by the denominator, and then subtracting from both sides the new right hand side (all steps which are reversible), we obtain

$$\left(\begin{array}{l} q_1 \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi^U(\{f\}) \mu_{\ell, e}^U(\{f\}) \\ \mu_{\ell, x}^M(\llbracket \varphi \wedge \text{pre}(f) \rrbracket^M \cap S_{\ell, x}^M \cap \pi_1(\llbracket \psi \rrbracket^{M[U]})) \\ -q \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi^U(\{f\}) \mu_{\ell, e}^U(\{f\}) \\ \mu_{\ell, x}^M(\llbracket \varphi \wedge \text{pre}(f) \rrbracket^M \cap S_{\ell, x}^M) \end{array} \right) \geq 0.$$

Then as $\pi_1(\llbracket \psi \rrbracket^{M[U]}) = \bigcup_{f \in X^U} \llbracket \text{pre}(f) \wedge [U, f]\psi \rrbracket^M$, we have that x is in

$$\left[\left[\begin{array}{l} q_1 \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi^U(\{f\}) \mu_{\ell, e}^U(\{f\}) \\ \text{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f]\psi) \\ -q \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi^U(\{f\}) \mu_{\ell, e}^U(\{f\}) \\ \text{P}_\ell(\varphi \wedge \text{pre}(f)) \end{array} \right] \geq 0 \right]^M.$$

By adding qt to both sides we see that this is equivalent to x being in

$$\left[\left[\begin{array}{l} q_1 \sum_{\varphi \in \Phi} \sum_{f \in X^U} \mu_\varphi(f) \mu_{\ell, e}(f) \\ \text{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f]\psi_k) \end{array} \right] \geq qt \right]^M,$$

which completes the proof of the soundness of this case.

OTHER AXIOMS AND RULES The soundness of the remaining rules is standard following the lead of [9, 10]. \dashv

Theorem 4.2. *The inference system given in Figure 1 is weakly complete for validity of \mathcal{L}_{UL} .*

To prove completeness, we reduce the problem of completeness of the proof system for \mathcal{L}_{UL} with respect to the semantics in Section 3 to the better known completeness problem of probabilistic epistemic logic \mathcal{L}_{PEL} in [3]. We thus translate \mathcal{L}_{UL} into \mathcal{L}_{PEL} in such a way that every $\varphi \in \mathcal{L}_{\text{UL}}$ is translated into a $t(\varphi) \in \mathcal{L}_{\text{PEL}}$, such that $\vdash \varphi \leftrightarrow t(\varphi)$.

To help with induction on formulas, we impose an order on formulas using the following complexity function, adapted from [2, 8, 7].

Definition 4.3 (Complexity). We define a complexity function $c : \mathcal{L}_{\text{UL}} \rightarrow \mathbb{Z}_+$ from formulas to positive integers as follows:

$$\begin{aligned} c(a) &\stackrel{\text{def}}{=} 1 \text{ for } a \in \text{AtProp} \cup \{\text{false}, \text{true}\} \\ c(\varphi \star \psi) &\stackrel{\text{def}}{=} 1 + \max\{c(\varphi), c(\psi)\} \text{ for } \star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\} \\ c(\neg\varphi) &\stackrel{\text{def}}{=} 1 + c(\varphi) \\ c([\ell]\varphi) &\stackrel{\text{def}}{=} 1 + c(\varphi) \\ c([U, e]\psi) &\stackrel{\text{def}}{=} (2 + |X^U| + \max\{c(\text{pre}(f)) \mid f \in X^U\}) \\ & \quad \cdot 2c(\psi) \\ c(\sum_{k=1}^n a_k \text{P}_\ell(\varphi_k) \geq b) &\stackrel{\text{def}}{=} \max\{c(\varphi_k) + 1 \mid 1 \leq k \leq n\} \end{aligned}$$

Theorem 4.4. *The function c from Definition 4.3 satisfies each of the following.*

1. If ψ is a proper subformula of φ , then $c(\psi) < c(\varphi)$.
2. $c(\text{pre}(e) \rightarrow q) < c([U, e]q)$ for $q \in \text{AtProp} \cup \{\text{true}, \text{false}\}$.
3. $c([U, e]\varphi \star [U, e]\psi) < c([U, e](\varphi \star \psi))$ for $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.
4. $c(\text{pre}(e) \rightarrow \neg[U, e]\varphi) < c([U, e]\neg\varphi)$.
5. $c(\text{pre}(e) \rightarrow \bigwedge_{f: eR_\ell^U f} [\ell][U, f]\varphi) < c([U, e][\ell]\varphi)$ for $\ell \in \mathbb{I}$.

$$\begin{aligned}
6. & c((t = 0 \wedge (\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \text{P}_\ell(\text{false}) \geq q)) \vee (\neg(t = 0) \wedge (\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \sum_{\varphi \in \Phi, f \in E} \mu_\varphi(f) \mu_{e,\ell}(f) \text{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f] \psi_k) \geq qt))) \\
& < c([U, e] \sum_{k=1}^n q_k \text{P}_\ell(\psi_k)), \text{ where } t = \sum_{\varphi \in \Phi, f \in E} \mu_\varphi(f) \mu_{e,\ell}(f) \text{P}_\ell(\varphi \wedge \text{pre}(f)).
\end{aligned}$$

7. If $c(\varphi) < c(\psi)$, then $c([U, e]\varphi) < c([U, e]\psi)$.

Proof This proof follows standard techniques, and we will only provide a proof of the most surprising item, namely item 6:

$$\begin{aligned}
& c((t = 0 \wedge (\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \text{P}_\ell(\text{false}) \geq q)) \vee (\neg(t = 0) \wedge (\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \sum_{\varphi \in \Phi, f \in E} \mu_\varphi(f) \mu_{e,\ell}(f) \text{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f] \psi_k) \geq qt))) < c([U, e] \sum_{k=1}^n q_k \text{P}_\ell(\psi_k)), \\
& \text{ where } t = \sum_{\varphi \in \Phi, f \in E} \mu_\varphi(f) \mu_{e,\ell}(f) \text{P}_\ell(\varphi \wedge \text{pre}(f)).
\end{aligned}$$

Proceeding, we have the following argument:

$$\begin{aligned}
& c((t = 0 \wedge (\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \text{P}_\ell(\text{false}) \geq q)) \vee (\neg(t = 0) \wedge (\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \sum_{\varphi \in \Phi, f \in E} \mu_\varphi(f) \mu_{e,\ell}(f) \text{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f] \psi_k) - qt \geq 0))) \\
& = 2 + \max\{c(t = 0), c(\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \text{P}_\ell(\text{false}) \geq q), c(\neg t = 0), c(\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \sum_{\varphi \in \Phi, f \in E} \mu_\varphi(f) \mu_{e,\ell}(f) (\text{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f] \psi_k) - q \text{P}_\ell(\varphi \wedge \text{pre}(f)) \geq 0))\} \\
& = 4 + \max_{k, \varphi, f}\{c(\varphi \wedge \text{pre}(f) \wedge [U, f] \psi_k), c(\varphi \wedge \text{pre}(f))\} \\
& = 6 + \max_{k, f}\{c([U, f] \psi_k)\} \\
& = 6 + |X^U| + \max_{k, f}\{(2 + c(\text{pre}(f))) \cdot 2c(\psi_k)\} \\
& < (2 + |X^U| + \max_f c(\text{pre}(f))) \cdot (2 \max_k \{c(\psi_k)\} + 2) \\
& = (2 + |X^U| + \max c(\text{pre}(f))) \cdot 2c(\sum_{k=1}^n q_k \text{P}_\ell(\psi_k) \geq q) \\
& = c([U, e] \sum_{k=1}^n q_k \text{P}_\ell(\psi_k) \geq q)
\end{aligned}$$

There are different values for the complexity of an update formula $[U, s]\psi$ used in the literature. The one given in [8] sets $c([U, e]\psi) = (4 + \max_f c(\text{pre}(f)))c(\psi)$. We add a multiplicity of 2 in order to handle the probabilistic update case in Theorem 4.4. But this also allows us to reduce the 4 to a 2 in all the cases, even the non-probabilistic ones.

We define the following translation function adapted from [7]:

$ \begin{aligned} t(q) &= q \text{ for } q \in \Phi \cup \{\text{true}, \text{false}\} \\ t(\neg\varphi) &= \neg t(\varphi) \\ t(\varphi_1 \star \varphi_2) &= t(\varphi_1) \star t(\varphi_2) \text{ for } \star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\} \\ t([\ell]\varphi) &= [\ell]t(\varphi) \\ t(\sum_{k=1}^n q_k \text{P}_\ell(\varphi_k) \geq q) &= \sum_{k=1}^n q_k \text{P}_\ell(t(\varphi_k)) \geq q \\ t([U, e]q) &= t(\text{pre}(e) \rightarrow q) \text{ for } q \in \Phi \cup \{\text{true}, \text{false}\} \\ t([U, e]\neg\varphi) &= t(\text{pre}(e) \rightarrow \neg[U, e]\varphi) \\ t([U, e](\varphi_1 \star \varphi_2)) &= t([U, e]\varphi_1 \star [U, e]\varphi_2) \\ &\quad \text{for } \star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\} \\ t([U, e][\ell]\varphi) &= t(\text{pre}(e) \rightarrow \bigwedge_{f: e R_\ell f} [\ell][U, f]\varphi) \end{aligned} $
$ \begin{aligned} t([U, e] \sum_{k=1}^n q_k \text{P}_\ell(\varphi_k) \geq q) &= \\ t(\tau = 0 \wedge (\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \text{P}_\ell(\text{false}) \geq q)) \vee \\ \neg(\tau = 0) \wedge (\text{pre}(e) \rightarrow \sum_{k=1}^n q_k \sum_{\varphi \in \Phi, f \in E} \mu_\varphi(f) \mu_{e,\ell}(f) \text{P}_\ell(\varphi \wedge \text{pre}(f) \wedge [U, f] \psi_k) \geq q\tau) \\ &\text{ where } \\ \tau &= \sum_{\varphi \in \Phi, f \in E} \mu_\varphi(f) \mu_{e,\ell}(f) \text{P}_\ell(\varphi \wedge \text{pre}(f)) \end{aligned} $
$t([U, e][U', f]\varphi) = t([U, e]t([U', f]\varphi))$

Proposition 4.5. *The function t is well-defined.*

Proof We prove this by a straightforward induction on $c(\varphi)$ that if $c(\varphi) \leq n$, then $t(\varphi)$ is well-defined. The base cases where $n = 1$ are immediate. For the inductive case, from Theorem 4.4, we see that the complexity of each formula that t is applied to on the left hand side of the definition is strictly greater than the complexity of each corresponding formula t is applied to on the right hand side. As for the case $c(t([U, e][U', f]\varphi)) > c(t([U, e]t([U', f]\varphi)))$, the key observation is that $c([U', f]\varphi) > c(t([U', f]\varphi))$, which can be obtained by observing each case based on what the main connective of φ is. \dashv

In preparation for showing that $t(\varphi)$ and φ are provably equivalent, we have the following lemma.

Lemma 4.6. *If $\vdash \varphi_k \leftrightarrow \psi_k$ for $1 \leq k \leq n$, then $\vdash \sum_{k=1}^n q_k \text{P}_\ell(\varphi_k) \geq q \leftrightarrow \sum_{k=1}^n q_k \text{P}_\ell(\psi_k) \geq q$, for all $q_1, \dots, q_n, q \in \mathbb{Q}$.*

Proof By the rule EQV, we have $\vdash \text{P}_\ell(\varphi_k) - \text{P}_\ell(\psi_k) \geq 0 \wedge \text{P}_\ell(\psi_k) - \text{P}_\ell(\varphi_k) \geq 0$, for $1 \leq k \leq n$.

Then by IA, IP, and I0,

$$\begin{aligned}
& \vdash \sum_{k=1}^n q_k \mathsf{P}_\ell(\varphi_k) \geq q \\
& \leftrightarrow \sum_{k=1}^n q_k \mathsf{P}_\ell(\varphi_k) + \sum_{k=1}^n [\mathsf{P}_\ell(\psi_k) - \mathsf{P}_\ell(\varphi_k)] \geq q + 0 \\
& \vdash \sum_{k=1}^n q_k \mathsf{P}_\ell(\varphi_k) + \sum_{k=1}^n [\mathsf{P}_\ell(\psi_k) - \mathsf{P}_\ell(\varphi_k)] \geq q + 0 \\
& \leftrightarrow \sum_{k=1}^n q_k \mathsf{P}_\ell(\psi_k) \geq q.
\end{aligned}$$

with the desired result following by propositional logic. \dashv

Theorem 4.7. *For each φ , $\vdash t(\varphi) \leftrightarrow \varphi$.*

Proof We prove by induction on $c(\varphi)$ that whenever $c(\varphi) \leq n$, $\vdash t(\varphi) \leftrightarrow \varphi$. The base cases with $c(\varphi) = 1$ are immediate. The boolean cases follow from propositional reasoning. The epistemic case comes from modal reasoning. The probabilistic case uses Lemma 4.6. The update cases are immediate from the close relationship between the definition of t and the axiom schema, as well as the fact that the induction hypothesis is immediately applicable. The complexity of an update formula decreases with each application of the reduction, since by Theorem 4.4, the formula that t is applied to on the left hand side is strictly greater in complexity than the formula that t is applied to on the right hand side. The update case $[U, e][U', f]\varphi$ makes use of the rule $[U, e]$ -NEC. \dashv

We finally show that $t(\varphi)$ has the desired form.

Theorem 4.8. *For each φ , $t(\varphi)$ contains no occurrences of update modalities.*

Proof This follows from a straightforward induction on $c(\varphi)$ that whenever $c(\varphi) \leq n$, then $t(\varphi)$ has no update modalities. It uses the fact that the formulas that t are applied to on the left side of the definition of t have strictly larger complexity than each corresponding formula on the right hand side that t is applied to. It also uses the fact that the Boolean, modal, and probability cases do not introduce update operators. \dashv

Completeness Model Construction

As any formula in \mathcal{L}_{UL} is provably equivalent to a formula in \mathcal{L}_{PEL} , in order to show completeness of our system for \mathcal{L}_{UL} , it is sufficient to show that every consistent formula $\theta \in \mathcal{L}_{\text{PEL}}$ is satisfiable in a probabilistic epistemic model. This follows from the well established fact that a system is complete with respect to

a semantics if and only if every consistent formula is satisfiable in a model (see for example [1]). There is essentially one difference between our semantics and the one in [3]: our models allow for the 0 measure as well as the probability measure, and this difference only plays a minor role near the end of the proof.

Here is a sketch of the general strategy for proving completeness, following the general strategy of [3]. We start with a consistent formula $\theta \in \mathcal{L}_{\text{PEL}}$, and let Δ be the set of its subformulas and negations of subformulas. (Note that Δ is finite.) We define a *canonical model* as $M = (X, R, V, \mathbb{P})$, where

- X is the set of maximally consistent subsets of Δ ,
- xRy iff for all $[\ell]\varphi \in \Delta$, $[\ell]\varphi \in x$ iff $\varphi \in y$,
- $V(p) = \{x \in X \mid p \in x\}$, and
- $\mathbb{P}_{\ell, x} = (X, \mathcal{P}(X), \mu_{\ell, x})$, where $\mu_{\ell, x}$ is either any function satisfying constraints that will be given later in Figure 2 if there is such a satisfying function or the 0 function in the case that there is no satisfying function.

Let

$$\{t_1 \geq r_1, \dots, t_k \geq r_k\}$$

be an enumeration of the inequality formulas in x , and let

$$\{t_{k+1} \geq r_{k+1}, \dots, t_m \geq r_m\}$$

be the inequality formulas in $\Delta \setminus x$.

Enumerate $X = \{x_1, \dots, x_n\}$. For each $x \in X$, let $\hat{x} = \bigwedge_{\psi \in x} \psi$. For each $\ell \in \mathbb{I}$, each formula $t_i \geq r_i$ is equivalent to $\sum_{j=1}^n a_{i,j} \mathsf{P}_\ell(\hat{x}_j) \geq r_i$ for appropriate coefficients $a_{i,j}$.⁴ Then the formula $t_i \geq r_i$ is true at x if and only if $\sum_{j=1}^n a_{i,j} \mu_{\ell, x}(x_j) \geq r_i$. We then have the system of inequalities given in Figure 2. Let $\mu_{\ell, x}$ be any satisfying function of this system if there is one (in which case $\mu_{\ell, x}$ is a probability function). If there is no such satisfying function, then that must be because it is provable from \hat{x} that $\neg \mathsf{P}_\ell(\text{true}) = 1$, in which case it is provable from \hat{x} that $\mathsf{P}_\ell(\text{true}) = 0$ using P2. Since x is consistent, the 0 function must then be compatible with the first m inequalities. This is really because the proof system (particularly the inequality axioms together with modus ponens) can follow the steps of a mathematical algorithm that determines whether there is a solution to a system of inequalities, as mentioned in [3].

Completeness follows from a truth lemma that says that for every formula $\varphi \in \Delta$ and every state $x \in X$,

$$\varphi \in x \Leftrightarrow x \in \llbracket \varphi \rrbracket^M.$$

⁴This is proved in [3] and relies on the additivity axiom P3, the rule EQV, and the provability of the formula $\mathsf{P}_\ell(\text{false}) = 0$, but not of the formula $\mathsf{P}_\ell(\text{true}) = 1$.

$\sum_{j=1}^n a_{1,j} \mu_{\ell,x}(y_j) \geq r_1$
\vdots
$\sum_{j=1}^n a_{k,j} \mu_{\ell,x}(y_j) \geq r_k$
$\sum_{j=1}^n a_{k+1,j} \mu_{\ell,x}(y_j) < r_{k+1}$
\vdots
$\sum_{j=1}^n a_{m,j} \mu_{\ell,x}(y_j) < r_m$
$\sum_{j=1}^n \mu_{\ell,x}(y_j) \geq 1$
$-\sum_{j=1}^n \mu_{\ell,x}(y_j) \geq -1$
$\mu_{\ell,x}(y_1) \geq 0$
\vdots
$\mu_{\ell,x}(y_n) \geq 0$

Figure 2: The system of inequalities

This is proved by induction on the formula, and is similar to the proof of the truth lemma for basic modal logic (see [1]). Note that the case for the probability formulas $t \geq r$ does not make use of the induction hypothesis, but follows directly from the fact that the chosen measure $\mu_{\ell,x}$ (be it a probability measure or the 0 function) is compatible with the first m inequalities.

5 Conclusion

This paper has focused on technical aspects of extending probabilistic dynamic epistemic logic from a finite setting to an infinite setting. In particular, we saw that by extending prior and posterior probabilities from a finite setting to an infinite setting, we are able to preserve the property of the semantics that every formula has a measurable denotation and we are able to establish a complete proof system. Extending observation and occurrence probabilities to an infinite setting however results in difficulties, including one discussed in [9], in which dynamics does not preserve the measurability of the denotation of the formulas; however the use of outer measure extensions alleviates the need for the formulas to denote measurable sets. The infinite setting promises a greater variety of applications that may be useful in, for example, robotics, where there is uncertainty in measurements over infinitely many possible values. We, however, leave such applications to future work. Other interesting directions of research include the study of the complexity of the logics, possible connections with the rich body of the theory of coalgebraic modal logics, and the development of model checking tools for this logic.

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