A coalgebraic approach to graded modal logic and graded bisimilarity

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This talk is based on joint work with Luca Aceto and Anna Ingolfsdottir

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Graded modal language

The graded modal language $\mathcal{L}_{\mathrm{gml}}$ is defined by

$$\varphi ::= \mathsf{true} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond_n \varphi \quad (n \in \mathbb{N})$$

We derive $\Box_n \varphi \equiv \neg\Diamond_n \neg\varphi$, and have the following basic readings:

- $\Diamond_n \varphi$ reads "φ has weight at least n."
- $\Box_n \varphi$ reads "$\neg\varphi$ has weight less than n."

What is meant by "weight" depends on different semantics, which we now define.

Semantics of graded modal logic

There are many choices of semantics. We focus on three

1. standard frame-semantics
   (on Kripke frames)
2. multi-frame semantics
   (on multi-frames, a type of weighted frames)
3. predicate-lifting coalgebraic semantics
   (on coalgebras that are equivalent to multi-frames)

Graded modal logic

Definition (Kripke Frame)

A Kripke frame is a tuple $F = (S, R)$, such that

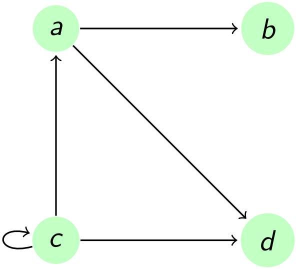
- S is a set (of "states")
- $R \subseteq S^2$ is a binary relation

A pair (F, s), $s \in S$ is a *pointed Kripke frame*.

- $(F, s) \models \Diamond_n \varphi$ if and only if
  there are at least n s-successors t, such that $(F, t) \models \varphi$.
  (φ holds in at least n-successor states.)
- $(F, s) \models \Box_n \varphi$ if and only if
  for all sets A of n s-successors, there is a $t \in A$ such that $(F, t) \models \varphi$.
  (φ fails to hold in fewer than n-successor states.)

Note that $\Box_1$ and $\Diamond_1$ coincide with the standard modal $\Box$ and $\Diamond$.

Frame semantics example



- $a \models \Diamond_2 \text{true} \wedge \Box_3 \neg \text{true} \wedge \Box_1 \Box_1 \neg \text{true}$ is read
  "There are at least 2 successors, fewer than 3 successors, and every successor from here has no successor."
- $c \models \Diamond_2 \Diamond_2 \text{true} \wedge \Diamond_1 \Box_1 \neg \text{true}$
  "There are at least 2 successors with at least 2 successors and there is at least one successor with no successors."

Multi-frames and weighted modal logic

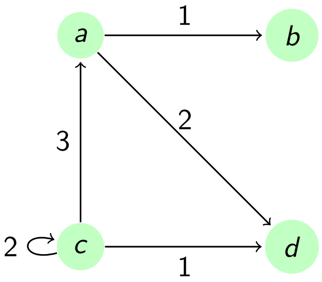
Definition (Multiframe (aka multigraph))

A multi-frame is a pair $M = (S, \Sigma)$, such that

- S is a set (lets assume it is countable)
- Σ consists of, for each $s \in S$, a multiset function $\sigma^s : S \to \mathbb{N}$ (here $\mathbb{N} = \{0, 1, 2, \dots\}$).

- $(M, s) \models \Diamond_n \varphi$ if and only if $\sum\{\sigma^s(t) \mid (M, t) \models \varphi\} \geq n$.
- $(M, s) \models \Box_n \varphi$ if and only if $\sum\{\sigma^s(t) \mid (M, t) \models \neg\varphi\} < n$.

Multi-frame semantics example



- $a \models \Diamond_3\,\mathsf{true} \wedge \Box_4 \neg\,\mathsf{true} \wedge \Box_1 \Box_1 \neg\,\mathsf{true}$ is read
  "The weighted out-degree is at least 3, and fewer than 4, and every successor from here has no successor."
- $c \models \Diamond_2 \Diamond_2\,\mathsf{true} \wedge \Diamond_1 \Box_1 \neg\,\mathsf{true}$
  "There is a weight of at least 2 of states whose weighted out-degree is at least 2 and there is at least one successor with no successors."

Translations between frame and multi-frame

Definition (Translation $\mathcal{M}$)

Given a Kripke frame $F = (S, R)$, we can translate it into the multi-frame $\mathcal{M}(F) = (S, \Sigma)$, where for each $s \in S$,

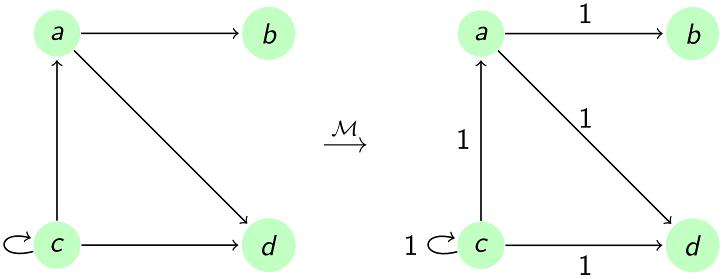
$$\sigma^s(t) = \begin{cases} 1 & sRt \\ 0 & \text{otherwise} \end{cases}$$

Definition (Translation $\mathcal{K}$)

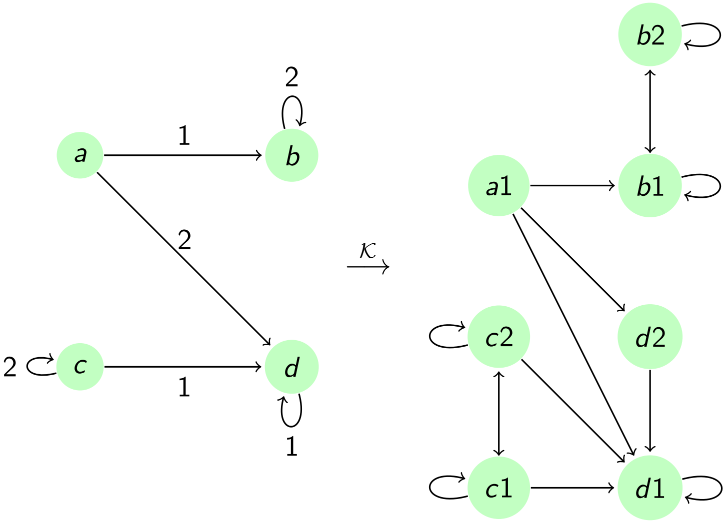
Given a multi frame $M = (S, \Sigma)$, we can translate it into a Kripke frame $\mathcal{K}(M) = (T, R)$, where

- $T = \bigcup_{s \in S} \{(s, n) \mid 1 \leq n \leq \sup_{a \in S}\{1, \sigma^a(s)\}\}$
- $(s, n)R(t, m)$ if and only if $1 \leq m \leq \sigma^s(t)$.

Example of translation $\mathcal{M}$



Example of translation $\mathcal{K}$



Relationship between frame and multi-frame

For any pointed frame or multiframe P, let

$$L(P) = \{\varphi \in \mathcal{L}_{\mathrm{gml}} \mid P \models \varphi\}.$$

Then

Given a pointed Kripke frame (F, s)

$$L(F, s) = L(\mathcal{M}(F), s).$$

Given a pointed multi-frame (M, s)

$$L(M, s) = L(\mathcal{K}(M), (s, 1)).$$

These are proved by a straightforward induction on formulas.

Coalgebraic modal logic background

Definition (Finite powerset functor)

The finite powerset functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ is given by

- For each object (set) X, $\mathcal{P}(X)$ consists of all subsets of X
- For each morphism (function) $f : X \to Y$, $\mathcal{P}f : Z \mapsto f[Z]$ for each finite subset $Z \subseteq X$.

Definition ($\mathcal{P}$-coalgebra)

A $\mathcal{P}$ coalgebra is a pair (A, α), where A is a set and $\alpha : X \to \mathcal{P}X$ is a morphism (function).

Each $\mathcal{P}$-coalgebra defines a Kripke frame (directed graph).

Predicate lifting background

Definition (Contravariant powerset functor)

The contravariant power set functor $2 : \mathsf{Set} \to \mathsf{Set}$ is given by

- For each X, $2(X) = \mathcal{P}(X)$ the set of all subsets of X
- For each morphism $f : X \to Y$, $2f : Z \mapsto f^{-1}[Z]$ for each subset Z of Y.

Natural transformation for contravariant functors

A natural transformation from contravariant functor F to G on Set, is a collection of morphisms λ_X for each set X, such that whenever $f : X \to Y$, the following commutes.

$$\begin{array}{ccc} FY & \xrightarrow{F(f)} & FX \\ {\scriptstyle \lambda_X}\downarrow & & \downarrow{\scriptstyle \lambda_Y} \\ GY & \xrightarrow[G(f)]{} & GX \end{array}$$

Predicate lifting semantics for powerset functor

Definition

A *predicate lifting* for $\mathcal{P}$ is a natural transformation $\lambda : 2 \to 2 \circ \mathcal{P}$, where 2 is the contravariant powerset functor.

For each set S, we define

$$\lambda_S : A \mapsto \{B \in \mathcal{P}(S) \mid A \cap B \neq \emptyset\}.$$

Given a coalgebra $X = (S, \alpha)$ and $s \in S$,

$$(X, s) \models \Diamond\varphi \text{ if and only if } \alpha(s) \in \lambda_S[\![\varphi]\!],$$

where $[\![\varphi]\!] = \{t \mid (X, t) \models \varphi\}$.

Multi-set functor

Definition (Finite mulit-set functor)

The finite multiset functor $\mathcal{B} : \mathrm{Set} \to \mathrm{Set}$ is given by

- For each set X, $\mathcal{B}(X)$ consists of all finite multisets on X (functions $\sigma : X \to \mathbb{N}$ with finite support).
- For each morphism f, $\mathcal{B}f\sigma : y \mapsto \sum\{\sigma(x) \mid f(x) = y\}$. $((\mathcal{B}f\sigma)(y)$ gives σ's "weighted sum" $\sigma(f^{-1}[y])$ of $f^{-1}[y])$

Definition ($\mathcal{B}$-coalgebra)

A $\mathcal{B}$ coalgebra is a pair (A, α), where A is a set and $\alpha : X \to \mathcal{B}X$ is a morphism (function).

Each $\mathcal{B}$-coalgebra defines a multiframe (aka multigraph).

Predicate lifting (semantics)

Definition

A *predicate lifting* for $\mathcal{B}$ is a natural transformation $\lambda : 2 \to 2 \circ \mathcal{B}$, where 2 is the contravariant powerset functor.

For each set S and $n \in \mathbb{N}$, we define

$$\lambda_S^n : A \mapsto \left\{ \sigma \in \mathcal{B}(S) \,\middle|\, \sum_{x \in A} \sigma(x) \geq n \right\}.$$

Given a coalgebra $X = (S, \alpha)$ and $s \in S$,

$$(X, s) \models \Diamond_n \varphi \text{ if and only if } \alpha(s) \in \lambda_S^n [\![\varphi]\!],$$

where $[\![\varphi]\!] = \{t \mid (X, t) \models \varphi\}$.

Relationship between multi-frame and coalgebra

Let

- $\mathcal{M}$ map each coalgebra X to its multiframe $\mathcal{M}(X)$, and
- $\mathcal{C}$ map each multi-frame M to its coalgebra $\mathcal{C}(M)$.

Then

Given a pointed coalgebra (X, s)

$$L(X, s) = L(\mathcal{M}(X), s).$$

Given a pointed multiframe (M, s)

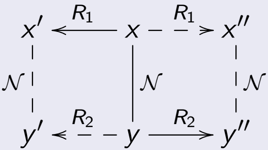
$$L(M, s) = L(\mathcal{C}(M), s).$$

Bisimulation: basic definition

Definition (Bisimulation)

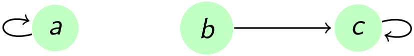
A bisimulation between $F_1 = (S_1, R_1)$ and $F_2 = (S_2, R_2)$ is a relation $\mathcal{N} \subseteq S_1 \times S_2$, such that whenever $x \mathcal{N} y$,

1. whenever $x R_1 x'$, there is a $y' \in R_2(y)$, such that $x' \mathcal{N} y'$.
2. whenever $y R_2 y''$, there is an $x'' \in R_1(x)$, such that $x'' \mathcal{N} y''$.



The largest bisimulation, denoted $\leftrightarroweq$, is called *bisimilarity*

Examples concerning bisimulations



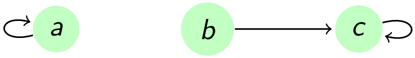
Every point is bisimilar to each other. In particular, $a \mathrel{\underline{\leftrightarrow}} b$.

No point is bisimilar to a distinct other. In particular, $x \not\mathrel{\underline{\leftrightarrow}} y$.

Aside comment

However, x and y are *mutually similar*, in that each simulates the other.

Examples concerning bisimulations



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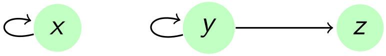

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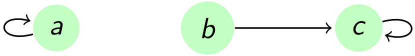

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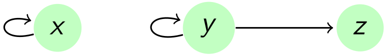

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However, x and y are *mutually similar*, in that each simulates the other.

Graded bisimulation

Definition (Graded bisimulation)

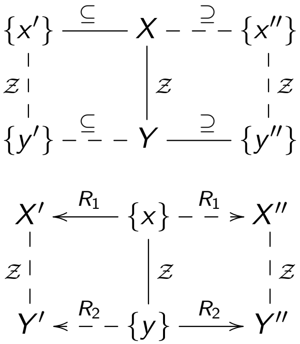
A g-bisimulation relation between $F_1 = (S_1, R_1)$ and $F_2 = (S_2, R_2)$ is a relation $\mathcal{Z} \subseteq \mathcal{P}^{<\omega}(S_1) \times \mathcal{P}^{<\omega}(S_2)$ satisfying

1. whenever $X \mathcal{Z} Y$,
   1. $|X| = |Y|$,
   2. for each $x \in X$, there is a $y \in Y$ such that $\{x\} \mathcal{Z} \{y\}$, and
   3. for each $y \in Y$, there is an $x \in X$ such that $\{x\} \mathcal{Z} \{y\}$;
2. whenever $\{x\} \mathcal{Z} \{y\}$
   1. if $X \subseteq R_1(x)$ is finite, then there exists some finite $Y \subseteq R_2(y)$ such that $X \mathcal{Z} Y$, and
   2. if $Y \subseteq R_2(y)$ is finite, then there exists some finite $X \subseteq R_1(x)$ such that $X \mathcal{Z} Y$.

The largest g-bisimulation, written $\leftrightarrows_g$, is called *g-bisimilarity*.

We will in general focus on *image-finite* Kripke frames.

Graded bisimulation in pictures



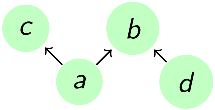
Relationship between bisimulation and g-bisimulation

Given any graded bisimulation $\mathcal{Z}$, the set

$$\mathcal{N} = \{(x, y) \mid (\{x\}, \{y\}) \in \mathcal{Z}\}$$

is a bisimulation.

The converse need not be true:



- $\mathcal{N} = \{b, c\}^2 \cup \{a, d\}^2$ is a bisimulation
- a and d are not g-bisimilar

Resource bisimulation

Definition (Resource bisimulation)

An r-bisimulation between $F_1 = (S_1, R_1)$ and $F_2 = (S_2, R_2)$ is a relation $\mathcal{R} \subseteq S_1 \times S_2$ satisfying

- whenever $(s_1, s_2) \in \mathcal{R}$, there is a bijective function

  $$f_{s_1,s_2} : R_1(s_1) \to R_2(s_2)$$

  such that

  $$(s, f_{s_1,s_2}(s)) \in \mathcal{R}$$

  for each $s \in R_1(s_1)$.

The largest r-bisimulation, written $\leftrightarroweq_r$, is called *r-bisimilarity*.

Relationship between g and r bisimulation

Given an r-bisimulation $\mathcal{R}$, there exists a g-bisimulation $\mathcal{Z}$, such that

$$\{(\{x\},\{y\}) \in \mathcal{Z} \mid (x,y) \in \mathcal{R}\}.$$

In particular,

$$\mathcal{Z} = \bigcup_{x\mathcal{R}y} \{(\{x\},\{y\})\} \cup \bigcup_{A \in \mathcal{P}^{<\omega}(R_1(x))} \{(A, f_{x,y}(A))\}.$$

Conversely, is it the case that if $\mathcal{Z}$ is a g-bisimulation, then $\{(x,y) \mid (\{x\},\{y\}) \in \mathcal{Z}\}$ is an r-bisimulation?

But what about g-bisimilarity and r-bisimilarity?

Goal

We aim to answer the second question.

Relationship between g and r bisimulation

Given an r-bisimulation $\mathcal{R}$, there exists a g-bisimulation $\mathcal{Z}$, such that

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Conversely, is it the case that if $\mathcal{Z}$ is a g-bisimulation, then $\{(x,y) \mid (\{x\},\{y\}) \in \mathcal{Z}\}$ is an r-bisimulation?

But what about g-bisimilarity and r-bisimilarity?

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Goal

We want, for any pointed Kripke frames (F, x) and (K, y), that

$$(F, x) \leftrightarroweq_r (K, y)$$

iff

$$(F, x) \leftrightarroweq_g (K, y).$$

Hennessy-Milner Property for g-bisimilarity

Definition (Hennessy-Milner property)

A bisimilarity $\Leftrightarrow$ has the Hennessy-Milner property if

$$(F_1, s_1) \Leftrightarrow (F_2, s_2) \text{ if and only if } L(F_1, s_1) = L(F_2, s_2).$$

Theorem

g-bisimilarity satisfies the Hennessy-Milner Property on image-finite Kripke frames.

Proof sketch (right-to-left)

Define

$X_1 \mathcal{Z} X_2$ if and only if each of the following holds:

- $|X_1| = |X_2|$,
- for each $x_1 \in X_1$, there is $x_2 \in X_2$ such that $L(x_1) = L(x_2)$, and
- for each $x_2 \in X_2$, there is $x_1 \in X_1$ such that $L(x_1) = L(x_2)$.

Given $L(w_1) = L(w_2)$, we have that $\{w_1\} \mathcal{Z} \{w_2\}$.

Recall relationships among semantics

Connection between frame and multi-frame

Given a pointed Kripke frame (F, s)

$$L(F, s) = L(\mathcal{M}(F), s).$$

The pointed multi-frame $(\mathcal{M}(F), s)$ satisfies the same formulas as the pointed frame (M, s).

Connection between multi-frame and coalgebra

Given a pointed multiframe (M, s)

$$L(M, s) = L(\mathcal{C}(M), s).$$

The pointed coalgebra $(\mathcal{C}(M), s)$ satisfies the same formulas as the pointed multi-frame (M, s).

Progress so far

The following are equivalent

- $L(\mathcal{CM}(F), x) = L(\mathcal{CM}(K), y)$
- $L(\mathcal{M}(F), x) = L(\mathcal{M}(K), y)$ Basic Induction
- $L(F, x) = L(K, y)$. Basic Induction
- $(F, x) \leftrightarroweq_g (K, y)$. HM property

r-bisimulation on multiframes

An r-bisimulation between *multiframes* M_1 and M_2 is the first coordinate projection of an ordinary frame r-bisimulation $\mathcal{R}$ between $\mathcal{K}(M_1)$ and $\mathcal{K}(M_2)$ that has the following constraint:

$$\text{If } (x, n)\mathcal{R}(y, m), \text{ then } (x, n')\mathcal{R}(y, m').$$

We have an equivalent formulation on the next slide.

r-bisimulation on multiframes

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We have an equivalent formulation on the next slide.

r-bisimulation on multiframes definition

Definition (r-bisimulation on multiframes)

A *(multiframe) r-bisimulation* between $M_1 = (S_1, \Sigma_1)$ and $M_2 = (S_2, \Sigma_2)$ is a relation $\mathcal{R} \subseteq S_1 \times S_2$ such that

- whenever $(s_1, s_2) \in \mathcal{R}$, there is a bijective function

$$f_{s_1,s_2} : \bigcup_{s \in S_1} \{(s,n) \mid 1 \leq n \leq \sigma_1^{s_1}(s)\} \to \bigcup_{s \in S_2} \{(s,n) \mid 1 \leq n \leq \sigma_2^{s_2}(s)\},$$

  such that $s \mathcal{R} t$ whenever $f_{s_1,s_2}(s,n) = (t,m)$.

The largest r-bisimulation on multiframes, also written $\Leftrightarrow_r$, is also called *r-bisimilarity*.

Multi-frame r-bisimulation that is a function

A function $g : S_1 \to S_2$ is a multiframe r-bisimulation between $M_1 = (S_1, \Sigma_1)$ and $M_2 = (S_2, \Sigma_2)$ precisely when for all $(s_1, s_2) \in S_1 \times S_2$

- if $g(s_1) = s_2$,
- then for each $y \in S_2$,

$$\sigma_2^{s_2}(y) = \sum_{\{x | g(x)=y\}} \sigma_1^{s_1}(x) = \sigma_1^{s_1}(g^{-1}[y])$$

Relationships between r-bisimilarities

Given pointed *multiframes* (M_1, s_1) and (M_2, s_2)

$$(M_1, s_1) \leftrightarrows_r (M_2, s_2) \text{ iff } (\mathcal{K}(M_1), (s_1, 1)) \leftrightarrows_r (\mathcal{K}(M_2), (s_2, 1))$$

Given pointed *Krikpe frames* (F_1, s_1) and (F_2, s_2)

$$(F_1, s_1) \leftrightarrows_r (F_2, s_2) \text{ iff } (\mathcal{M}(F_1), s_1) \leftrightarrows_r (\mathcal{M}(F_2), s_2)$$

Progress so far

The following are equivalent

- $(F, x) \leftrightarrows_r (K, y)$.
- $(\mathcal{M}(F), x) \leftrightarrows_r (\mathcal{M}(K), y)$ — Direct argument

The following are equivalent

- $L(\mathcal{CM}(F), x) = L(\mathcal{CM}(K), y)$
- $L(\mathcal{M}(F), x) = L(\mathcal{M}(K), y)$ — Basic Induction
- $L(F, x) = L(K, y)$. — Basic Induction
- $(F, x) \leftrightarrows_g (K, y)$. — HM property

Coalgebraic homomorphism

Definition (homomorphism)

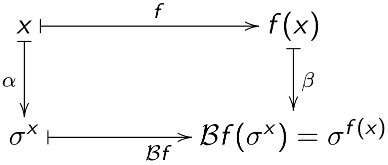
A map f from (A, α) to (B, β) is a homomorphism if the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ FX & \xrightarrow[Ff]{} & FY \end{array}$$

If $F = \mathcal{P}$, then *coalgebraic homomorphism* coincides with bounded morphism on frames (a function that is a bisimulation).

Equivalent multiframe homomorphism

If $F = \mathcal{B}$, then we examine point by point:



Lifting σ^x additively to sets, f is a coalgebraic homomorphism iff for each $x \in X$, $\beta(f(x)) = \sigma^{f(x)}$ is given by

$$\sigma^{f(x)} : y \mapsto \sigma^x(f^{-1}(y)).$$

The coalgebraic morphism is a function that is a *multiframe resource bisimulation*.

Coalgebraic bisimulation

Let $F : \mathrm{Set} \to \mathrm{Set}$.

Definition (Coalgebraic bisimulation)

A relation $R \subseteq A \times B$ is a coalgebraic bisimulation between F-coalgebras (A, α) and (B, β) if there is a morphism $\delta : R \to FR$, such that the following commutes:

$$
\begin{array}{ccccc}
A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\
\alpha \downarrow & & \downarrow \delta & & \downarrow \beta \\
FA & \xleftarrow{F\pi_1} & FR & \xrightarrow{F\pi_2} & FB
\end{array}
$$

where $\pi_1 : R \to A$ and $\pi_2 : R \to B$ are projections that are F-coalgebra morphisms.

States $x \in A$ and $y \in B$ are *bisimilar*, $x \leftrightarrows_c y$, if there exists a bisimulation R between (A, α) and (B, β), such that $(x, y) \in R$.

Relationship between bisimulation and coalg. bisimulation

A coalgabraic bisimulation is a bisimulation

If $F = \mathcal{P}^{<\infty}$ or $F = \mathcal{B}$, then F-morphisms are bisimulations. Hence $R = \pi_1 \circ \pi_2^{-1}$ is the composition of bisimulations and thus a bisimulation.

A bisimulation is a coalgebraic bisimulation

- If $F = \mathcal{P}$ and R is a bisimulation, then δ exists:

$$\delta : (x, y) \mapsto (\alpha(x) \times \beta(y)) \cap R.$$

- If $F = \mathcal{B}$ and R is a multiframe resource bisimulation, then R is a coalgebraic bisimulation via a matrix property (for each $(a, b) \in R$, $\delta(a, b)(x, y)$ is given by the (x, y) coordinate of a $|X| \times |Y|$ matrix.)

Coalgebraic bisimilarity and resource bisimilarity

Definition (Matrix property of a relation)

Let (A, Σ) and (B, T) be multiframes. A relation $\mathcal{R} \subseteq A \times B$ satisfies the matrix property if for every $(a, b) \in \mathcal{R}$, there exists an $|A| \times |B|$-matrix $(m_{x,y})$ with entries from $\mathbb{N}$ such that

1. all but finitely many $m_{x,y}$ are 0,
2. $m_{x,y} \neq 0$ implies $(x, y) \in \mathcal{R}$,
3. for each x, $\sigma^a(x) = \sum\{m_{x,y} \mid y \in B\}$, and
4. for each y, $\tau^b(y) = \sum\{m_{x,y} \mid x \in A\}$.

- R is a multiframe r-bisimulation iff matrix property holds
  $m_{x,y} > 0$ is the number of pairs $((x, n), (y, m))$, such that $f_{a,b} : (x, n) \mapsto (y, m)$.
- R is a coalgebraic bisimulation iff matrix property holds
  $\delta(a, b) : (x, y) \mapsto m_{x,y}$

Progress so far

The following are equivalent

- $(F, x) \leftrightarroweq_r (K, y)$.
- $(\mathcal{M}(F), x) \leftrightarroweq_r (\mathcal{M}(K), y)$ — Direct argument
- $(\mathcal{C}\mathcal{M}(F), x) \leftrightarroweq_c (\mathcal{C}\mathcal{M}(K), y)$ — Matrix property

The following are equivalent

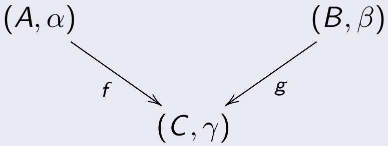
- $L(\mathcal{C}\mathcal{M}(F), x) = L(\mathcal{C}\mathcal{M}(K), y)$
- $L(\mathcal{M}(F), x) = L(\mathcal{M}(K), y)$ — Basic Induction
- $L(F, x) = L(K, y)$. — Basic Induction
- $(F, x) \leftrightarroweq_g (K, y)$. — HM property

Coalgebraic behavioral equivalence

Let $F : \mathrm{Set} \to \mathrm{Set}$.

Definition (Behavioural equivalence)

Pointed F-coalgebras $((A, \alpha), x)$ and $((B, \beta), y)$ are behaviorally equivalent, written $((A, \alpha), x) \leftrightarroweq_b ((B, \beta), y)$, if there exist a coalgebra (C, γ), and coalgebra homomorphisms $f : A \to C$ and $g : B \to C$, such that $f(x) = g(y)$.



Bisimulation and Behavioral Equivalence

Definition (Final coalgebra)

A final coalgebra is a coalgebra (A, α), such that for every coalgebra (B, β) there is a unique coalgebraic homomorphism φ from (B, β) to (A, α).

The finite multiset functor $\mathcal{B}$ has a final coalgebra.

Theorem

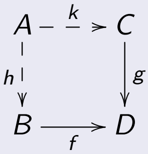
If two pointed coalgebras are bisimilar, then they are behaviorally equivalent, so long as a final coalgebra exists.

For the functor $\mathcal{B}$, if two two models are coalgebraically bisimilar then they are behavioral equivalent.

Preserving weak pullbacks

Definition (Weak pullback)

Given functions $f : B \to D$ and $g : C \to D$, a weak pullback is a pair of functions $h : A \to B$ and $k : A \to C$, such that $g \circ k = h \circ f$ and whenever $f(b) = g(c)$ for some $b \in B$ and $c \in C$, there exists an $a \in A$ such that $h(a) = b$ and $k(a) = c$. This is depicted by a diagram called a *weak pullback square*:



Definition (Preserving weak pullbacks)

A functor F : Set $\to$ Set *preserves weak pullbacks* if the image of a weak pullback square under F is also a weak pullback square.

Finite multiset functor preserves weak pullbacks

The key step in proving that the Finite multiset functor preserves weak pullbacks is to apply the following:

Theorem (Row-column theorem (integer version))

if $p_1, \ldots, p_m, q_1, \ldots, q_n \in \mathbb{N}$ *are such that* $\sum_{1\leq i\leq m} p_i = \sum_{1\leq j\leq n} q_j$, *then for each* $1 \leq i \leq m$ *and* $1 \leq j \leq n$, *there exists* $r_{ij} \in \mathbb{N}$, *such that*

$$\sum_{1\leq j\leq n} r_{ij} = p_i, \ \mathit{for}\ 1 \leq i \leq m, \ \mathit{and} \sum_{1\leq i\leq m} r_{ij} = q_j, \ \text{for}\ 1 \leq j \leq n.$$

Coalgebraic bisimilarity and behavioral equivalence

Theorem

Behavioral equivalence implies bisimilarity when the functor preserves weak pullbacks.

Progress so far

The following are equivalent

- $(F, x) \leftrightarroweq_r (K, y)$.
- $(\mathcal{M}(F), x) \leftrightarroweq_r (\mathcal{M}(K), y)$ — Direct argument
- $(\mathcal{CM}(F), x) \leftrightarroweq_c (\mathcal{CM}(K), y)$ — Matrix property
- $(\mathcal{CM}(F), x) \leftrightarroweq_b (\mathcal{CM}(K), y)$ — $\Uparrow$ weak pullbacks; $\Downarrow$ final coalg

The following are equivalent

- $L(\mathcal{CM}(F), x) = L(\mathcal{CM}(K), y)$
- $L(\mathcal{M}(F), x) = L(\mathcal{M}(K), y)$ — Basic Induction
- $L(F, x) = L(K, y)$. — Basic Induction
- $(F, x) \leftrightarroweq_g (K, y)$. — HM property

Behavioral equivalence and formulas

Definition (Separating)

A set $\{\lambda^n_X\}_{n\in\mathbb{N}}$ of predicate liftings is *separating* if, for every set X, any multiset $\sigma \in \mathcal{B}(X)$ can be uniquely determined by the set

$$\mathbb{S}(\sigma) = \{(\lambda^n_X, A) \mid n \in \mathbb{N}, A \subseteq X, \sigma \in \lambda^n_X(A)\}.$$

Our set $\{\lambda^n_X\}_{n\in\mathbb{N}}$ is separating, since given $x \in X$ and $\sigma : X \to \mathbb{N}$,

$$\mathbb{X} = \{(\lambda^n_X, \{x\}) \mid \sigma \in \lambda^n_X(\{x\})\} = \{(\lambda^n_X, \{x\}) \mid n \leq \sigma(x)\} \subseteq \mathbb{S}(\sigma)$$

is enough to recover σ (by $\sigma = x \mapsto \max\{n \mid (\lambda^n_X, \{x\}) \in \mathbb{X}\}$).

Theorem

If $\mathcal{L}$ is a $\mathcal{B}$-coalgebraic modal logic using a separating set of predicate liftings, then the logic is expressive (that is, if $L(X, s) = L(Y, t)$, then $(X, s) \leftrightarrows_b (Y, t)$).

homomorphisms preserving semantics

Definition

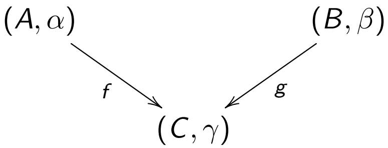
Homomorphisms preserve the semantics if for every homomorphism $f : X \to Y$ (with $X = (A, \alpha)$, $Y = (B, \beta)$), formula φ, and $a \in A$,

$$(X, a) \models \varphi \text{ if and only if } (Y, f(a)) \models \varphi.$$

A simple induction shows that homomorphisms preserve our semantics.

behavioral equivalence preserves semantics

Suppose $((A, \alpha), a) \leftrightarrows_b ((B, \beta), b)$. Then the following diagram



commutes, and $f(a) = g(b)$.

As homomophisms preserve the semantics,

$$L((A, \alpha), a) = L((C, \gamma), f(a)) = L((B, \beta), b).$$

Outline of proof

The following are equivalent

- $(F, x) \leftrightarroweq_r (K, y)$.
- $(\mathcal{M}(F), x) \leftrightarroweq_r (\mathcal{M}(K), y)$ — Direct argument
- $(\mathcal{CM}(F), x) \leftrightarroweq_c (\mathcal{CM}(K), y)$ — Matrix property
- $(\mathcal{CM}(F), x) \leftrightarroweq_b (\mathcal{CM}(K), y)$ $\Uparrow$ Weak pullbacks; $\Downarrow$ Final coalg
- $L(\mathcal{CM}(F), x) = L(\mathcal{CM}(K), y)$ $\Uparrow$ Separating; $\Downarrow$ Hom pres sem
- $L(\mathcal{M}(F), x) = L(\mathcal{M}(K), y)$ — Basic Induction
- $L(F, x) = L(K, y)$. — Basic Induction
- $(F, x) \leftrightarroweq_g (K, y)$. — HM property

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THANK YOU!