# Refined inversion statistics on permutations

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# Permutations and Inversions

**Definition**

A permutation of rank $n$ is a bijective function $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

**Definition**

An inversion in a permutation $\pi$ is a pair $(a, b)$, such that $0 \leq a < b \leq n$ and $\pi_a > \pi_b$.
A non-inversion is a pair $(a, b)$, such that $0 \leq a < b \leq n$ and $\pi_a < \pi_b$.

**Example**

$\pi = 32415$ has

- four inversions: $(1, 2)$, $(1, 4)$, $(2, 4)$, $(3, 4)$
- six non-inversions: $(1, 3)$, $(1, 5)$, $(2, 3)$, $(2, 5)$, $(3, 5)$, $(4, 5)$

# Non-inversion Sums

$\mathrm{INV}(\pi)$ is the set of all inversions of $\pi$ and
$\mathrm{NINV}(\pi)$ is the set of all non-inversions of $\pi$.

## Definition

- The inversion sum of $\pi$ is given by

$$\mathrm{invsum}(\pi) = \sum_{(a,b)\in \mathrm{INV}(\pi)} (b-a)$$

- The non-inversion sum of $\pi$ is given by

$$\mathrm{ninvsum}(\pi) = \sum_{(a,b)\in \mathrm{NINV}(\pi)} (b-a)$$

## Proposition

$$\sum_{(a,b)\in \mathrm{NINV}(\pi)} \pi(b)-\pi(a) = \sum_{(a,b)\in \mathrm{NINV}(\pi)} b-a$$

# Cosine of a permutation

Let $\mathbf{1}$ be the identity permutation of rank $n$.

**Definition**

For any permutation $\pi$ of rank $n$, the cosine of $\pi$ is

$$\cos(\pi) = \mathbf{1} \cdot \pi = \sum_{i=1}^{n} i\pi(i)$$

**Observation**

Given a permutation $\pi$ of rank $n$, if $\theta$ is the angle between the vectors corresponding to $\pi$ and $\mathbf{1}$.

$$\cos(\pi) = a(n) \cos(\theta),$$

where $a(n) = n(n+1)(2n+1)/6$.

# Cosine Theorem

**Theorem**

*For $k \geq 35$, there exists a permutation $\pi$ such that* $\cos(\pi) = \mathbf{1} \cdot \pi = k$

The total number of permutations $\pi$ such that $\cos(\pi) = k$ is given by the sequence

A135298 in the *Online Encyclopedia of Integer Sequences*.

Our theorem shows that this sequence is non-zero after $k = 34$.

# Non-inversion sum and cosine

If $\pi = \pi_1 \cdots \pi_n$, then $\pi^{\mathrm{c}} = (n+1-\pi_1)\cdots(n+1-\pi_n)$.

**Theorem**

$$\cos(\pi) = \mathbf{1}\cdot\mathbf{1}^{\mathrm{c}} + \mathrm{ninvsum}(\pi) = \binom{n+2}{3} + \mathrm{ninvsum}(\pi)$$

**Theorem**

*For $n \geq 4$ and $0 \leq k \leq \binom{n+1}{3}$, there is a permutation $\pi$ such that* $\mathrm{ninvsum}(\pi) = k$.

**Theorem**

*For $n \geq 6$,*

$$\binom{n+1}{3} + \binom{n}{3} \geq \binom{n+2}{3} - 1$$

# Zone-crossing vector

**Definition**

Given a permutation $\pi$ of rank $n$, its non-inversion zone-crossing vector is $\mathrm{nzcv}(\pi) = (z_1, z_2, \ldots, z_n)$, where $z_k$ is the number of non-inversions $(a, b) \in \mathrm{NINV}(\pi)$, where $a \leq k < b$.

**Proposition**

The sum of the coordinates of $\mathrm{nzcv}(\pi)$ is equal to $\mathrm{ninvsum}(\pi)$.

# Distribution function for zone-crossing vector coordinates

Let $\mathrm{nzcv}(\pi)_k$ be the $k^{\mathrm{th}}$ coordinate of $\mathrm{nzcv}(\pi)$.

**Theorem**

*The number of permutations of rank n, such that* $\mathrm{nzcv}(\pi)_k = j$ *is*

$$k!(n-k)![q^j]\begin{bmatrix} n \\ k \end{bmatrix}_q,$$

*where for any polynomial* $p(q)$*, its coefficient of* $q^j$ *is denoted by* $[q^j]p(q)$*, and*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad [n]_q! = \prod_{k=1}^{n} \frac{1-q^k}{1-q}.$$

# Distribution function for non-inversion sum

We are interested in the distribution function for non-inversion sums:

$$N_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathsf{ninvsum}(\pi)}$$

**Theorem**

$$N_{n+1}(q) = N_n(q) + \sum_{k=1}^{n-1} q^{\binom{k+1}{2}} \sum_{\pi \in \mathfrak{S}_n} q^{\mathsf{nzcv}(\pi)_k} q^{\mathsf{ninvsum}(\pi)} + q^{\binom{n+1}{2}} N_n(q).$$

# THANK YOU!


- J. Sack, H. Ulfarsson. *Refined inversion statistics on permutions*. arXiv:1106.1995, 2011