Quantum Logic and Structure

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Two perspectives on quantum logic

Express testable properties

- Testable properties may be ranges of values of a particle's position, momentum, or velocity.
- Organize testable properties as a Hilbert lattice: the lattice of closed-linear subspaces of a Hilbert spaces

Express quantum dynamics

- Dynamics resulting from two kinds of actions: unitary evolutions and quantum tests
- Model the effects of actions using a labelled transition system

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Original Quantum Logic: focus on testable properties

The original quantum logic helps us reason about relationships among testable properties:

1. p, q denote testable properties (closed linear subspaces)
2. $p \wedge q$ the intersection of two closed subspaces
3. $p \vee q$ the closure of the span of two closed subspaces
4. $\neg p$ the orthocomplement of a closed subspace

The testable properties form a lattice with a complement.

$p \leq q$ $\qquad$ p is at least as strong a property as q

Founding paper on quantum logic:

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Basic results

Dunn, Hagge, Moss, Wang (2005)

Quantum logic over finite dimensional Hilbert spaces is decidable

Decision procedures rely on computations involving real numbers.

Hermann & Ziegler (2011)

1. The 1 and 2 dimensional satisfiability problems are NP-complete (same as classical Boolean logic)
2. The n dimensional satisfiability problem is complete for Non-deterministic Blum-Shub-Smale computation for $n \geq 3$ (random access to registers that contain real values)

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An essential property of a Hilbert lattice

Hilbert lattices are bounded and satisfy

1. $p^{\perp\perp} = p$;
2. $p \leq q$ implies $q^{\perp} \leq p^{\perp}$;
3. $p \wedge p^{\perp} = O$ and $p \vee p^{\perp} = I$.

A lattice with these properties is called an ortholattice, and its logic orthologic.

The negation of orthologic is "classical" as opposed to intuitionistic

Distributivity and Weak Modularity

Hilbert lattices are not distributive: consider x, y, z non-equal one-dimensional subspaces of $\mathbb{R}^2$.

1. $x \vee (y \wedge z) \neq (x \vee y) \wedge (x \vee z)$
2. $x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z)$.

But Hilbert lattices do satisfy weak modularity (aka orthomodularity):

$$q \leq p \;\Rightarrow\; p[q] = q,$$

where $p[q] := p \wedge (p^{\perp} \vee q)$.

Weak modularity adds some distributivity to the lattice.

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Projection and lattice dynamics

Projection is an action arising from a quantum test.

Sasaki Projection $p[q] := p \wedge (p^{\perp} \vee q)$
The result of projecting q onto p.

Weak modularity ensures that $p[-]$ is idempotent.

Sasaki Hook $[p]q := p^{\perp} \vee (p \wedge q)$
The precondition of a projection onto p resulting in q.

Additional significance of Weak Modularity

Proposition (Coecke and Smets 2004)

In an ortholattice $\mathfrak{L} = (L, \leq, -^{\perp})$ satisfies weak modularity if and only if $p[-]$ is a left adjoint of $[p]-$, that is

$$p[-] \dashv [p]-$$

In the context of Hilbert lattices, this is the equivalence of

1. The projection of x is contained in y
2. x is contained the precondition of projecting onto y.

- Coecke and Smets. The Sasaki hook is not a [static] implicative connective but induces a backward [in time] dynamic one that assigns causes. *International Journal of Theoretical Physics*, 2004.

Completeness and Atomicity

A Hilbert lattice is complete:
(Closed linear subspaces are closed under arbitrary intersections)

A Hilbert lattice is atomic:
(Every positive-dimensional subspace contains a one-dimensional subspace)

Projections of atoms

Covering Law

If a is an atom and $a \wedge p = O$, then $a \vee p$ covers p.

In an orthomodular lattice, this is equivalent to

If a is at atom and $a \not\leq p^{\perp}$, then $p[a]$ is an atom.

One dimensional subspaces project onto one-dimensional subspaces.

Propositional System

A propositional system is an orthoattice with all the properties mentioned
(weak modularity, completeness, atomicity, covering law)

Hilbert geometries are projective geometries with an orthogonality operator

A categorical equivalence has been established between propositional systems and Hilbert geometries.

- Stubbe and van Steirteghem. Propositional systems, Hilbert lattices, and generalized Hilbert spaces. Handbook of Quantum Logic and Quantum Structures, 2007.

Superposition Principle

For *distinct* atoms a and b, there is an atom c distinct from the others, such that $a \vee c = b \vee c = a \vee b$.

$a \vee b$ is a two-dimensional space. Each atom is spanned by the other two.

Proposition

For a propositional system, the following are equivalent:

- the Superposition Principle holds;
- The lattice is irreducible.

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Piron lattice

A Piron lattice is a *propositional system* satisfying the *Superposition Principle*.

A generalized Hilbert space (aka orthomodular vector space) extends the notion of Hilbert space to modules over division rings.

A Piron lattice with at least 4 mutually orthogonal atoms can be realized by a the bi-orthogonally closed subspaces of a generalized Hilbert space.

- Piron. *Foundations of Quantum Physics*. W.A. Benjamin Inc, 1976.

Ortholattice characterization of Hilbert spaces

A Piron lattice $\mathfrak{L} = (L, \leq, -^{\perp})$ is said to satisfy Mayet's condition if there is an automorphism $k : L \to L$ such that

1. there is a $p \in L$ such that $k(p) < p$, and
2. there is a $q \in L$ such that there are at least two distinct atoms below q and $k(r) = r$ for all $r \leq q$.

A Piron lattices satisfying Mayet's condition characterize the lattices of closed subspaces of infinite-dimensional Hilbert spaces over the reals, complex numbers, or quaternions.

- Mayet. Some Characterizations of the Underlying Division Ring of a Hilbert Lattice by Automorphisms. *International Journal of Theoretical Physics*, 1998.

Summary

Infinite dimensional Hilbert spaces over complex numbers, real numbers, or quaternions can be characterized lattice theoretically by the properties given earlier.

Finite dimensional Hilbert spaces are harder to characterize, since there are more possible division rings underlying finite-dimensional orthomodular vector spaces.

Labelled transition systems and dynamics

A basic labelled transition system consists of

1. A set S of states
2. A set A of actions
3. Relations $\xrightarrow{a} \subseteq S \times S$ for each $a \in A$

$s \xrightarrow{a} t$ means that action a can transform state s into state t.

Logics, such as Hennesey Milner logic and propositional dynamic logic, that are interpreted on labelled transition systems are have been widely used for reasoning about *classical computation*.

For every action a there is a model operator $[a]$ in the language s.t. for any formula ϕ

$$s \models [a]\phi \Leftrightarrow t \models \phi \text{ whenever } s \xrightarrow{a} t$$

Quantum computation

1. States are one dimensional subspaces (atoms of the Hilbert lattice)
2. Actions include
   - Projections onto testable properties
   - Unitary operations

Probabilistic Logic for Quantum Actions

We use the labelled transition system setting to reason about quantum computation.

Probabilistic Quantum Dynamic Logic

A decidable probabilistic quantum dynamic logic that can express quantum algorithms, such as Grover's Search algorithm.

- Baltag, Bergfeld, Kishida, Sack, Smets, Zhong. PLQP & Company: Decidable Logics for Quantum Algorithms. International Journal of Theoretical Physics, 2014.

Let

- N be a fixed finite subset of the natural numbers.
- U be a set of symbols for unitary operators.

The language is two-sorted (where $I \subseteq N$, $r \in \mathbb{Q}$, $u \in U$):

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid [\pi]\varphi \mid K_I\varphi \mid P^{\geq r}\varphi$$

$$\pi ::= u \mid \varphi? \mid \pi;\pi \mid \pi \cup \pi$$

- Basic propositional language
- Propositional dynamic logic language
- Probabilistic Language for Quantum Actions
- $K_I\varphi$ - a "knowledge" operator capturing properties local to a quantum subsystem
- $P^{\geq r}\varphi$ - probability that a test of φ succeeds is at least r.

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Probabilistic Hilbert Semantics

Fix a finite set N of numbers. Let $\mathcal{H} = \bigotimes_i \mathcal{H}_i$ be the tensor product of finite Hilbert spaces.

- States the one-dimensional subspaces of $\mathcal{H}$.
- Actions all possible terms $\pi ::= u \mid \varphi? \mid \pi;\pi \mid \pi \cup \pi$
  - Projection: For each formula ϕ,
    - $T = [\![\phi]\!]$ is a set of states (explained more on future slide),
    - $\overline{T}$ is the smallest closed subspace containing it, and
    - $[\![\varphi?]\!] = Proj_{\overline{T}}$ is the projection operator of vectors in $\mathcal{H}$ onto $\overline{T}$, acting on states.
  - Unitary operators: For each $u \in U$ fix a unitary operator $[\![u]\!]$ on $\mathcal{H}$ acting on states.
  - Sequential composition: $[\![\pi_1;\pi_2]\!] = [\![\pi_1]\!] \circ [\![\pi_2]\!]$ is relation composition
  - Choice: $[\![\pi_1 \cup \pi_2]\!] = [\![\pi_1]\!] \cup [\![\pi_2]\!]$

Interpretation of the formulas

For each formula φ, we define the set $[\![\varphi]\!]$ of states

- $[\![p]\!]$ is a closed subspace of $\mathcal{H}$.
- $[\![\neg\varphi]\!] = S \setminus [\![\varphi]\!]$ (set complement, not orthocomplement)
- $[\![\pi]\!]\varphi$ is the largest set of states T, such that $[\![\pi]\!][T] \subseteq \overline{[\![\varphi]\!]}$
- $[\![P^{\geq r}\varphi]\!]$ denotes the set of states whose non-zero vectors v are such that $\langle v \mid Proj_{\overline{[\![\varphi]\!]}} \mid v\rangle \geq r$.
- $[\![K_I\varphi]\!]$ denotes the set of states whose image under every I-remote unitary, $Id_I \otimes U_{N\setminus I}$, is in $[\![\varphi]\!]$.

Abbreviations

$$
\begin{aligned}
\bot &\overset{\text{def}}{=} \varphi \wedge \neg\varphi \\
\top &\overset{\text{def}}{=} \neg\bot \\
{\sim}\varphi &\overset{\text{def}}{=} [\varphi?]\bot \\
\varphi \sqcup \psi &\overset{\text{def}}{=} {\sim}({\sim}\varphi \wedge {\sim}\psi) \\
\varphi \vee \psi &\overset{\text{def}}{=} \neg(\neg\varphi \wedge \neg\psi) \\
\varphi \rightarrow \psi &\overset{\text{def}}{=} \neg(\varphi \wedge \neg\psi)
\end{aligned}
$$

$$
\begin{aligned}
\langle\pi\rangle\varphi &\overset{\text{def}}{=} \neg[\pi]\neg\varphi \\
\Diamond\varphi &\overset{\text{def}}{=} \langle\varphi?\rangle\top \\
\Box\varphi &\overset{\text{def}}{=} \neg\Diamond\neg\varphi \\
E\varphi &\overset{\text{def}}{=} \Diamond\Diamond\varphi \\
A\varphi &\overset{\text{def}}{=} \neg E\neg\varphi \\
\varphi_I &\overset{\text{def}}{=} \neg K_I\neg\varphi
\end{aligned}
$$

Quantum Search Example

$(N+1)$-qubit Quantum Search Problem

- Let $|f_0\rangle$ be a classical state of type **N**:

$$f_0 \in^{\mathbf{N}} \{0,1\} \text{ and } |f_0\rangle = \otimes_{i \in \mathbf{N}} |f_0(i)\rangle_i$$

- Let O be an action (“oracle”), such that

$$O(|f\rangle \otimes |b\rangle_N) = \begin{cases} |f\rangle \otimes |1-b\rangle_N, & \text{if } f = f_0, \\ |f\rangle \otimes |b\rangle_N, & \text{if } f \in {}^{\mathbf{N}}\{0,1\} \setminus \{f_0\} \end{cases}$$

- Goal: determine $|f_0\rangle$ from O.

Quantum Search Example

Quantum Search Algorithm

1. Input: $s = \otimes_{i=0}^{N} |a_i\rangle_i$, where $a_i = 0$ for $0 \le i \le N-1$ and $a_N = 1$.
2. Apply Hadamard gate to each qubit
3. Repeat the following K times (K largest integer less than $\pi\sqrt{2^N}/4$):
   1. Apply Oracle O to all qubits
   2. Apply Hadamard gate to all but last qubit
   3. Apply Conditional Phase Shift gate to all but last qubit.
   4. Apply Hadamard gate to all but last qubit
4. Measure (classical basis): let $|g\rangle$ be the resulting state of the first N qubits

Correctness Criterion

The algorithm is correct if $|g\rangle = |f_0\rangle$ with probability $p > 0.5$.

Classical state

An N-bit classical state is a function $f : N \to 2$, where $2 = \{0, 1\}$. Let

$$\underline{f} \stackrel{\text{def}}{=} \bigwedge_{i \in N} f(i)_i$$

$$CState(p) \stackrel{\text{def}}{=} Ep \wedge A(p \to \bigvee_{f : N \to 2} \underline{f})$$

Oracle

$$
\begin{aligned}
Ora(O) := \bigvee_{f:N\to 2} A\Big[& \big(\underline{f} \wedge 0_n \to [O](\underline{f} \wedge 1_n)\big) \\
& \wedge \big(\underline{f} \wedge 1_n \to [O](\underline{f} \wedge 0_n)\big) \\
& \wedge \bigwedge_{g:N\to 2, g\neq f} \Big(\big(\underline{g} \wedge 0_n \to [O](\underline{g} \wedge 0_n)\big) \\
& \qquad\qquad \wedge \big(\underline{g} \wedge 1_n \to [O](\underline{g} \wedge 1_n)\big)\Big)\Big].
\end{aligned}
$$

Quantum Search Algorithm Expressed

$$QSA := Ora(O) \wedge CState(p) \\ \wedge A\big(p \wedge 0_n \to [O]1_n\big) \wedge A\big(p \wedge 1_n \to [O]0_n\big) \wedge \underline{\mathbf{0}} \wedge 1_n \\ \to [H_0; \cdots ; H_n][O; H_0; \cdots ; H_{n-1}; P; H_0; \cdots ; H_{n-1}]^k P^{>0.5} p.$$

O is an oracle and p is a classical state

f is the state selected by the Oracle

After performing the operations and measurement, the result will more likely than not to be correct.

Returning to simplicity

Can we relate the quantum labelled transition systems, used for the probabilistic logic for quantum actions, to Hilbert lattices?

Quantum labelled transition systems

A dynamic frame is a tuple $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ such that

- Σ is a set (states of the system);
- $\mathcal{L} \subseteq \mathcal{P}(\Sigma)$ (testable properties);
- $\xrightarrow{P?} \subseteq \Sigma \times \Sigma$ for each $P \in \mathcal{L}$ (dynamics of tests);

Non-Orthogonality and Orthogonality

$s \to t \iff$ there is some $P \in \mathcal{L}$ such that $s \xrightarrow{P?} t$.

$s \not\to t \iff$ there is no $P \in \mathcal{L}$ such that $s \xrightarrow{P?} t$.

A dynamic frame is a quantum dynamic frame if it satisfies seven conditions in next slides.

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(1) Closure Condition

Closure Condition

$\mathcal{L}$ is closed under arbitrary intersection and orthocomplement, where the orthocomplement of $A \subseteq \Sigma$ is

$$\sim A := \{s \in \Sigma \mid s \to t \Rightarrow t \notin A, \forall t \in \Sigma\}.$$

Atomicity

For any $s \in \Sigma$, $\{s\} \in \mathcal{L}$.

Every singleton is in $\mathcal{L}$.

(3) Adequacy

Adequacy

For any $s \in \Sigma$ and $P \in \mathcal{L}$, if $s \in P$, then $s \xrightarrow{P?} s$.

The restriction of $\xrightarrow{P?}$ to P is reflexive.

(4) Repeatability

Repeatability

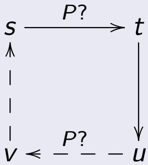
For any $s, t \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t$, then $t \in P$.

The image of $\xrightarrow{P?}$ is in P.

(5) Self-Adjointness

Self-Adjointness

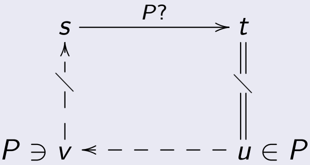
For any $s, t, u \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t \to u$, then there is a $v \in \Sigma$ such that $u \xrightarrow{P?} v \to s$.



(6) Covering Property

Covering Property

Suppose $s \xrightarrow{P?} t$ for $s, t \in \Sigma$ and $P \in \mathcal{L}$. Then, for any $u \in P$, if $u \neq t$ then $u \to v \not\to s$ for some $v \in P$.



(7) Proper Superposition

Proper Superposition

For any $s, t \in \Sigma$ there is a $u \in \Sigma$ such that $u \to s$ and $u \to t$.

$\to$ composed with $\to$ is the total relation.

Quantum Dynamic Frame (Summary)

1. **Closure Condition:** $\mathcal{L}$ is closed under arbitrary intersection and orthocomplement.
2. **Atomicity:** For any $s \in \Sigma$, $\{s\} \in \mathcal{L}$.
3. **Adequacy:** For any $s \in \Sigma$ and $P \in \mathcal{L}$, if $s \in P$, then $s \xrightarrow{P?} s$.
4. **Repeatability:** For any $s, t \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t$, then $t \in P$.
5. **Self-Adjointness:** For any $s, t, u \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t \rightarrow u$, then there is a $v \in \Sigma$ such that $u \xrightarrow{P?} v \rightarrow s$.
6. **Covering Property:** Suppose $s \xrightarrow{P?} t$ for $s, t \in \Sigma$ and $P \in \mathcal{L}$. Then, for any $u \in P$, if $u \neq t$ then $u \rightarrow v \not\rightarrow s$ for some $v \in P$.
7. **Proper Superposition:** For any $s, t \in \Sigma$ there is a $u \in \Sigma$ such that $u \rightarrow s$ and $u \rightarrow t$.

Mayet's condition on frames

A quantum dynamic frame automorphism is a bijective map $g : \Sigma \to \Sigma$ that preserves both orthogonality and non-orthogonality.

Definition (Mayet's condition for Quantum Dynamic Frames)

A quantum dynamic frame $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ is said to satisfy Mayet's condition if there is a strong automorphism $g : \Sigma \to \Sigma$ such that

1. there is a $P \in \mathcal{L}$ such that $g^{-1}[P] \subset P$, and
2. there is a $Q \in \mathcal{L}$ that has at least two distinct elements and such that $g(s) = s$ for all $s \in Q$.

Quantum dynamic frames were introduced by Baltag and Smets.

Baltag-Smets conjecture

Any quantum dynamic frame satisfying Mayet's condition can be realized by a Piron lattices with Mayet's condition.

- Baltag and Smets. Complete axiomatizations for quantum actions. *International Journal of Theoretical Physics*, 2005

Our result

Theorem

There is a categorical duality between

1. Piron lattices and quantum dynamic frames
2. Piron lattices with Mayet's condition and quantum dynamic frames with Mayet's condition

- Bergfeld, Kishida, Sack, and Zhong. Duality for the Logic of Quantum Actions. *Studia Logica*, 2015.

Concluding remarks

1. Richness of structure of testable properties
2. Connect quantum mechanical perspective (Hilbert spaces) with discrete (lattice) and computational (frame) perspectives
3. Exist quantum logics that describe probabilistic outcomes as well as composite systems (quantum entanglement)
4. Future work: Dualities between algebraic and relational quantum structures involving probabilities

Thank you!