The Probabilistic Logic of Communication and Change

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Urn example and Informational cascades

Example

- A room has either one of two urns:
  - Majority White: $\{W, W, B\}$
  - Majority Black: $\{B, B, W\}$.
- People line up to enter the room one-by-one to:
  - Draw a ball (observe and replace)
  - Write down a guess (for all to see) as to which urn it is

- In forming a guess, agents take into account:
  - The outcome (ball) of their draw
  - The guesses (which urn) that came before
- A Cascade develops if agents' conclusions are dominated by guesses that came before
  - False cascade: a cascade where agents' conclusions do not match the real situation
  - Correct cascade: a cascade where agents' conclusions do match the real situation

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Goal

We want a logic (to reason about cascades and related examples) with

- Probabilistic components (which urn is more likely)
- Epistemic components (beliefs agents have about each other)
- Common knowledge (of the rules of the example)
- Dynamic updates (to model how agents' views change)

Probabilistic Logic of Communication and Change

We propose the Probabilistic Logic of Communication and Change, the synthesis of variants of the following logics:

- Logic of Communication and Change

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This is a dynamic epistemic-PDL without knowledge axioms. We enforce knowledge axioms. Epistemic-PDL can express common knowledge.

- Dynamic Epistemic Probabilistic Logic

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Syntax

PLCC

Language of the Probabilistic Logic of Communication and Change:

$$\phi ::= true \mid p \mid \neg\phi \mid \phi \wedge \phi' \mid [\pi]\phi \mid [e]\phi \mid t_a \geq \beta \quad \text{(formulas)}$$

$$t_a ::= \alpha \cdot P_a(\phi) \mid t_a + t'_a \quad \text{(probability terms)}$$

$$\pi ::= a \mid \pi;\pi' \mid \pi \cup \pi' \mid \pi^* \mid ?\phi \quad \text{(epistemic programs)}$$

$p \in At$ (atomic proposition) such as the urn has majority white
$a \in Ag$ (agent)
$\alpha, \beta \in \mathbb{Q}$ (rational numbers)
$e \in E$ (epistemic update event) such as drawing from urn

Examples

- $.6 \cdot P_a(p) + .2 \cdot P_a([b]p) \geq .5$
- $[(a \cup b)^*]p$ It is common knowledge among a and b that p.
- $[e][a]p$ After informational event e, a would believe p.

Static sublanguage

PE-PDL

Language of Probabilistic Epistemic Propositional Dynamic Logic: Same as PLCC, but with dynamic formulas $[e]\varphi$ removed.

Models

Definition (Bayesian Kripke models)

$M = (S, \sim, \mu, V)$ where:

- S - (non-empty) set of states
- $\sim$ - equivalence relations $\sim_a$ on S, for each agent a
- μ - indexed probability functions $\mu_a : S \to (S \to [0,1])$, with values denoted $\mu_a^s(s')$, and satisfying:
  - State-determined probability (SDP)
    each agent knows her probability
    if $s \sim_a t$, then $\mu_a^s(s') = \mu_a^t(s')$ for all $s' \in S$
  - Consistency (CONS)
    consistency of probabilities with knowledge
    $\mu_a^s(t) = 0$ if $s \not\sim_a t$
  - Caution (CAUT) agents are cautious
    $s \not\sim_a t$ if $\mu_a^s(t) = 0$
  - Probability (PROB) μ_a^s is a probability mass function.
    $\sum_{t \in S} \mu_a^s(t) = 1$
- $V : At \to \mathcal{P}(S)$ a valuation function.

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Select semantics of PE-PDL

$$
\begin{array}{lll}
M, s \models p & \text{iff} & s \in V(p) \\
M, s \models [\pi]\phi & \text{iff} & M, t \models \phi \text{ whenever } sR_\pi t \\
M, s \models \sum_{j=1}^{n} \alpha_j P_a(\phi_j) \geq \beta & \text{iff} & \sum_{j=1}^{n} \alpha_j \cdot \mu_a^s(\phi_j) \geq \beta
\end{array}
$$

where $\mu_a^s(\phi_j)$ abbreviates $\sum_{s' \in S, s' \models \phi_j} \mu_a^s(s')$,

and R_π is a binary relation given by

$$
\begin{array}{lll}
sR_a t & \text{iff} & s \sim_a t \\
sR_{\pi_1 \cup \pi_2} t & \text{iff} & s(R_{\pi_1} \cup R_{\pi_2})t \\
sR_{\pi_1 ; \pi_2} t & \text{iff} & sR_{\pi_1} ; R_{\pi_2} t \ \ (\text{there is } w \text{, such that } sR_{\pi_1} w \text{ and } wR_{\pi_2} t) \\
sR_{\pi^*} t & \text{iff} & s(R_\pi)^* t \ \ ((R_\pi)^* \text{ the reflexive transitive closure of } R_\pi) \\
sR_{?\phi} t & \text{iff} & s = t \text{ and } M, s \models \phi
\end{array}
$$

PE-PDL example

Example (Relativized common knowledge)

Let

$$C_{\{a,b\}}(\psi,\phi) \stackrel{\text{def}}{=} [(?\psi;(a\cup b))^*]\phi$$

"ϕ is common knowledge among a and b conditional on ψ".

$M, s \models C_{\{a,b\}}(\psi,\phi)$ if and only if every path involving $\sim_a$ and $\sim_b$ from s that consists exclusively of states that make ψ true ends in a state that makes ϕ true.

Completeness of PE-PDL

Theorem

There is a sound and weakly complete proof system for PE-PDL with respect to Bayesian Kripke models.

- The proof system combines axioms for
  - agent programs (from propositional dynamic logic),
  - knowledge (for equivalence relations),
  - probabilities (and mix with atomic agent programs)
  - linear inequalities
- The completeness proof adapts the proof of PDL to one where atomic actions/agents are equivalence relations.

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$$[\pi](\phi \to \psi) \to ([\pi]\phi \to [\pi]\psi)$$
$$[\pi_1;\pi_2]\phi \leftrightarrow [\pi_1][\pi_2]\phi$$
$$[\pi_1 \cup \pi_2]\phi \leftrightarrow [\pi_1]\phi \wedge [\pi_2]\phi$$
$$[\pi^*]\phi \leftrightarrow (\phi \wedge [\pi][\pi^*]\phi)$$
$$[\pi^*](\phi \to [\pi]\phi) \to (\phi \to [\pi^*]\phi)$$
$$[\phi?]\psi \leftrightarrow (\phi \to \psi)$$

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$$[a]\phi \to \phi$$
$$[a]\phi \to [a][a]\phi$$
$$\neg[a]\phi \to [a]\neg[a]\phi$$

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    $P_a(\phi) \geq 0$

    $P_a(true) = 1$

    $P(\phi \wedge \psi) + P(\phi \wedge \neg\psi) = P_a(\phi)$

    $P_a(\phi) = P_a(\psi)$ if $\phi \leftrightarrow \psi$ is a propositional tautology.

    $[a]\phi \leftrightarrow P_a(\phi) \geq 1$

    $w \rightarrow [a]w$, for any w an a-probability formulas.
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$$t \geq \beta \leftrightarrow t + 0P_a(\phi) \geq \beta$$

$$\sum_{k=1}^{n} \alpha_k P_a(\phi_k) \geq \beta \rightarrow \sum_{k=1}^{n} \alpha_{j_k} P_a(\phi_{j_k}) \geq q\beta$$

(for any permutation $j_1, \ldots, j_n$ of $1, \ldots, n$)

$$\sum_{k=1}^{n} \alpha_k P_a(\phi_k) \geq \beta \wedge \sum_{k=1}^{n} \alpha'_k P_a(\phi_k) \geq \beta' \rightarrow \sum_{k=1}^{n} (\alpha_k + \alpha'_k) P_a(\phi_k) \geq (\beta + \beta')$$

$$t \geq \beta \leftrightarrow dt \geq d\beta \text{ if } d > 0$$

$$t \geq \beta \vee t \leq \beta$$

$$t \geq \beta \rightarrow t \geq \gamma \text{ if } \beta > \gamma$$

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Event model

Definition (Event Models)

$A = (E, \sim, \Phi, \text{pre}, \text{sub})$ where:

- E - the (finite non-empty) set of epistemic update events.
- $\sim$ - equivalence relations $\sim_a$ on E for each agent a.
- Φ - finite pairwise inconsistent set of precondition formulas
- pre - functions $\text{pre}_a : \Phi \to (E \to [0,1])$ for each $a \in Ag$, $\text{pre}_a(\phi)$ is a subjective occurrence probability function $(\sum_{e \in E} \text{pre}_a(\phi)(e) = 1)$

$$\text{PRE} : E \to \mathcal{P}(\Phi) \qquad \text{PRE} : e \mapsto \{\phi \mid \text{pre}(\phi)(e) > 0\}$$

$$\text{pre}_a(e \mid s) = \begin{cases} \text{pre}_a(\phi)(e) & \phi \in \Phi, M, s \models \phi \\ 0 & \text{there is no such } \phi \end{cases}$$

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Update product

Definition (Product Update)

$M = (S, \sim, \mu, V)$ Bayesian Kripke model
$A = (E, \sim, \Phi, \mathrm{pre}, \mathrm{sub})$ event model
$M \otimes A = (S \otimes E, \sim, \mu, V)$ update product

- $S \otimes E \stackrel{\text{def}}{=} \{(s, e) \mid s \in S, e \in E, (M, s) \models \bigvee \mathrm{PRE}(e)\}$.
- $(s, e) \sim_a (s', e')$ iff $s \sim_a s'$ and $e \sim_a e'$.
- Let $D \stackrel{\text{def}}{=} \sum_{(s',e') \sim_a (w,g)} (\mu_a^w(s') \cdot \mathrm{pre}_a(e' \mid s'))$, and put:

$$\mu_a^{(w,g)}(s, e) \stackrel{\text{def}}{=} \begin{cases} \frac{\mu_a^w(s) \cdot \mathrm{pre}_a(e|s)}{D} & \text{if } (s, e) \sim_a (w, g) \\ 0 & \text{otherwise} \end{cases}$$

  (Note that $D \neq 0$ for $(w, g) \in S \otimes E$.)
- $V(p) = \{(s, e) \mid M, s \models \mathrm{sub}(e)(p)\}$

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- Let $D \stackrel{\text{def}}{=} \sum_{(s',e') \sim_a (w,g)} (\mu_a^w(s') \cdot \mathrm{pre}_a(e' \mid s'))$, and put:

$$\mu_a^{(w,g)}(s, e) \stackrel{\text{def}}{=} \begin{cases} \frac{\mu_a^w(s) \cdot \mathrm{pre}_a(e \mid s)}{D} & \text{if } (s, e) \sim_a (w, g) \\ 0 & \text{otherwise} \end{cases}$$

  (Note that $D \neq 0$ for $(w, g) \in S \otimes E$.)
- $V(p) = \{(s, e) \mid M, s \models \mathrm{sub}(e)(p)\}$

The semantics of PLCC is the same as for PE-PDL, but with the following extra clause:

$$M, s \models [e]\phi \quad \text{iff} \quad M, s \models \bigvee \text{PRE}(e) \text{ then } M \times A, (s, e) \models \phi,$$

where e is an event in action model A

Urn example with formulas

Basic elements of the language

- $Ag \stackrel{\text{def}}{=} \{1, \ldots, n\}$ (agents)
- $At \stackrel{\text{def}}{=} \{\mathrm{MW}, \mathrm{MB}\} \cup \{\mathrm{DW}_i, \mathrm{DB}_i, \mathrm{W}_i, \mathrm{B}_i\}_{i \in Ag}$ (atoms)
- $E \stackrel{\text{def}}{=} \{\mathrm{dw}_i, \mathrm{db}_i, \mathrm{w}_i, \mathrm{b}_i\}_{i \in Ag}$ (events)

Some useful formulas and sets:

$$\chi_i \stackrel{\text{def}}{=} (\mathrm{MW} \vee \mathrm{MB}) \wedge \bigwedge_{j<i}(\mathrm{DW}_j \vee \mathrm{DB}_j) \wedge \bigwedge_{j<i}(\mathrm{W}_j \vee \mathrm{B}_j)) \wedge \bigwedge_{p \in At_{\geq i}} \neg p$$

The situation right before *i* draws

$$\chi_i^D \stackrel{\text{def}}{=} (\mathrm{MW} \vee \mathrm{MB}) \wedge \bigwedge_{j \leq i}(\mathrm{DW}_j \vee \mathrm{DB}_j) \wedge \bigwedge_{j<i}(\mathrm{W}_j \vee \mathrm{B}_j) \wedge \neg(\mathrm{W}_i \vee \mathrm{B}_i) \wedge \bigwedge_{p \in At_{>i}} \neg p$$

The situation right before *i* writes

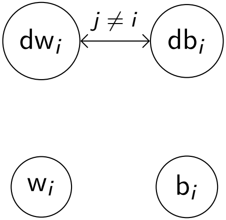
$$At_{\geq i} \stackrel{\text{def}}{=} \{\mathrm{DW}_j, \mathrm{DB}_j, \mathrm{W}_j, \mathrm{B}_j\}_{i \leq j \leq n}$$

$$At_{>i} \stackrel{\text{def}}{=} \{\mathrm{DW}_j, \mathrm{DB}_j, \mathrm{W}_j, \mathrm{B}_j\}_{i < j \leq n}$$

Depiction of Event Model: updates of probabilities

$\text{pre}(\psi_i^W)(\text{dw}_i) = \frac{2}{3}$
$\text{pre}(\psi_i^B)(\text{dw}_i) = \frac{1}{3}$

$\text{pre}(\psi_i^W)(\text{db}_i) = \frac{1}{3}$
$\text{pre}(\psi_i^B)(\text{db}_i) = \frac{2}{3}$



$\text{pre}(\phi_i^W)(\text{w}_i) = 1$
$\text{pre}(\phi_i^B)(\text{w}_i) = 0$

$\text{pre}(\phi_i^W)(\text{b}_i) = 0$
$\text{pre}(\phi_i^B)(\text{b}_i) = 1$

$\psi_i^W \stackrel{\text{def}}{=} \text{MW} \wedge \chi_i$

The urn has majority white and
it is right before agent i draws a ball.

$\psi_i^B \stackrel{\text{def}}{=} \text{MB} \wedge \chi_i$

$\phi_i^W \stackrel{\text{def}}{=} P_i(\text{MW}) > P_i(\text{MB}) \vee (\text{DW}_i \wedge P_i(\text{MW}) = P_i(\text{MB})) \wedge \chi_i^D$

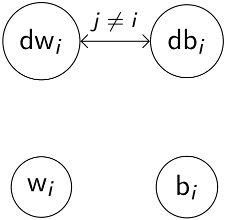
i either considers the majority white urn more likely or
i considers them equally likely but had just drawn white.

$\phi_i^B \stackrel{\text{def}}{=} P_i(\text{MB}) > P_i(\text{MW}) \vee (\text{DB}_i \wedge P_i(\text{MW}) = P_i(\text{MB})) \wedge \chi_i^D$

Depiction of Event Model: updates of atoms

$$\mathrm{sub}(\mathrm{dw}_i, p) = \begin{cases} \chi_i & p = \mathrm{DW}_i \\ p & p \neq \mathrm{DW}_i \end{cases}$$

$$\mathrm{sub}(\mathrm{db}_i, p) = \begin{cases} \chi_i & p = \mathrm{DB}_i \\ p & p \neq \mathrm{DB}_i \end{cases}$$



$$\mathrm{sub}(\mathrm{w}_i, p) = \begin{cases} \chi_i^D & p = \mathrm{W}_i \\ p & p \neq \mathrm{W}_i \end{cases}$$

$$\mathrm{sub}(\mathrm{db}_i, p) = \begin{cases} \chi_i^D & p = \mathrm{B}_i \\ p & p \neq \mathrm{B}_i \end{cases}$$

$$\chi_i \stackrel{\text{def}}{=} (\mathrm{MW} \vee \mathrm{MB}) \wedge \bigwedge_{j<i}(\mathrm{DW}_j \vee \mathrm{DB}_j) \wedge \bigwedge_{j<i}(\mathrm{W}_j \vee \mathrm{B}_j)) \wedge \bigwedge_{p \in At_{\geq i}} \neg p$$

The situation right before i draws

$$\chi_i^D \stackrel{\text{def}}{=} (\mathrm{MW} \vee \mathrm{MB}) \wedge \bigwedge_{j\leq i}(\mathrm{DW}_j \vee \mathrm{DB}_j) \wedge \bigwedge_{j<i}(\mathrm{W}_j \vee \mathrm{B}_j) \wedge \neg(\mathrm{W}_i \vee \mathrm{B}_i) \wedge \bigwedge_{p \in At_{>i}} \neg p$$

The situation right before i writes

$$At_{\geq i} \stackrel{\text{def}}{=} \{\mathrm{DW}_j, \mathrm{DB}_j, \mathrm{W}_j, \mathrm{B}_j\}_{i \leq j \leq n}$$

$$At_{>i} \stackrel{\text{def}}{=} \{\mathrm{DW}_j, \mathrm{DB}_j, \mathrm{W}_j, \mathrm{B}_j\}_{i < j \leq n}$$

Result

$$\chi \stackrel{\text{def}}{=} (\mathrm{MW} \vee \mathrm{MB}) \wedge \neg(\mathrm{MW} \wedge \mathrm{MB}) \wedge$$

Exactly one urn is placed in the room

$$\bigwedge_{i \in Ag} \big(P_i(\mathrm{MW}) = P_i(\mathrm{MB})\big) \wedge$$

Each agent considers each urn equally likely

$$\bigwedge_{p \in At_{\geq 1}} \neg p$$

Noone has drawn or written anything

Proposition

For all $1 \leq j \leq i$, *let* $f_j \in \{\mathrm{dw}_j, \mathrm{db}_j\}$ *and* $g_j \in \{\mathrm{w}_j, \mathrm{b}_j\}$. *Then*

$$[(\cup_{i \in Ag} i)^*]\chi \Rightarrow$$
$$[\mathrm{dw}_1][g_1][\mathrm{dw}_2][g_2][f_3][g_3] \ldots [f_i][g_i]\big(P_k(\mathrm{MW}) > P_k(\mathrm{MB})\big)$$

If it is common knowledge of χ, and the first two balls to be drawn are white, then after subsequent draws, every agent will consider the majority white urn to be more likely.

Result

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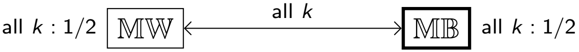
$$[\mathrm{dw}_1][g_1][\mathrm{dw}_2][g_2][f_3][g_3] \ldots [f_i][g_i]\left(P_k(\mathrm{MW}) > P_k(\mathrm{MB})\right)$$

If it is common knowledge of χ, and the first two balls to be drawn are white, then after subsequent draws, every agent will consider the majority white urn to be more likely.

Let

- $\mathbb{MW}$ be a state whose only atom is MW and
- $\mathbb{MB}$ be a state whose only atom is MB

Any model satisfying $[(\cup_{i \in Ag} i)^*]\chi$ and MB is bisimilar to:



Recall

$$\chi \stackrel{\text{def}}{=} (\text{MW} \vee \text{MB}) \wedge \neg(\text{MW} \wedge \text{MB}) \wedge$$

Exactly one urn is placed in the room

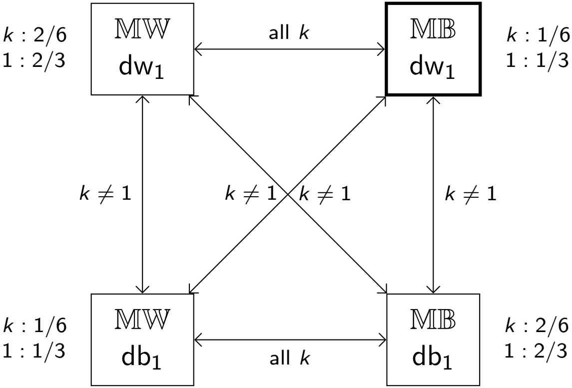
$$\bigwedge_{i \in Ag} \left(P_i(\text{MW}) = P_i(\text{MB})\right) \wedge$$

Each agent considers each urn equally likely

$$\bigwedge_{p \in At_{\geq 1}} \neg p$$

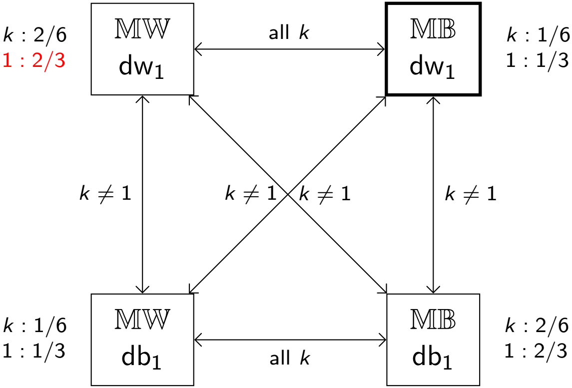
Noone has drawn or written anything

After agent 1 draws white



Agent 1 considers the majority white urn to be more likely, and will write white.

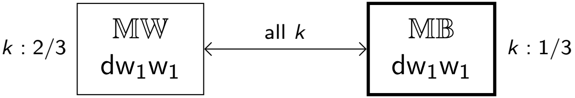
After agent 1 draws white



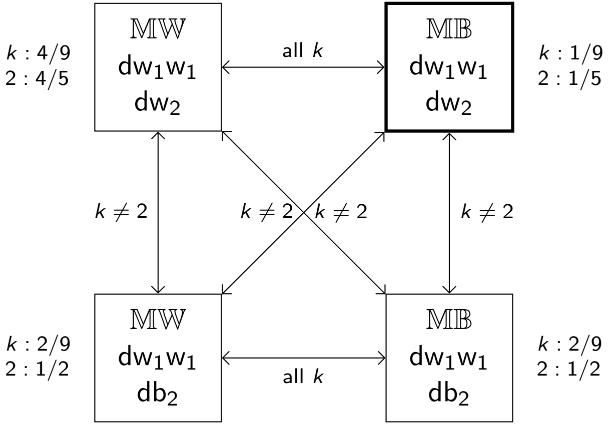
Agent 1 considers the majority white urn to be more likely, and will write white.

Agent 1 has just written her guess

The result of agent 1's guess is bisimilar (via generated submodel) to the following:

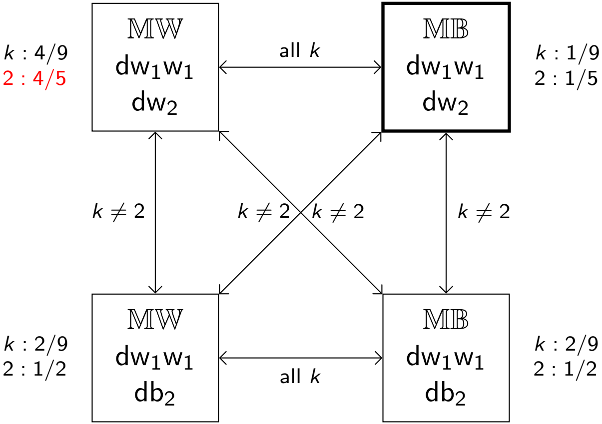


After second agent draws white



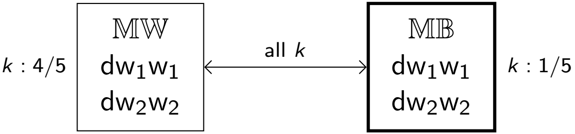
Agent 2 considers the majority white urn to be more likely, and will write white.

After second agent draws white

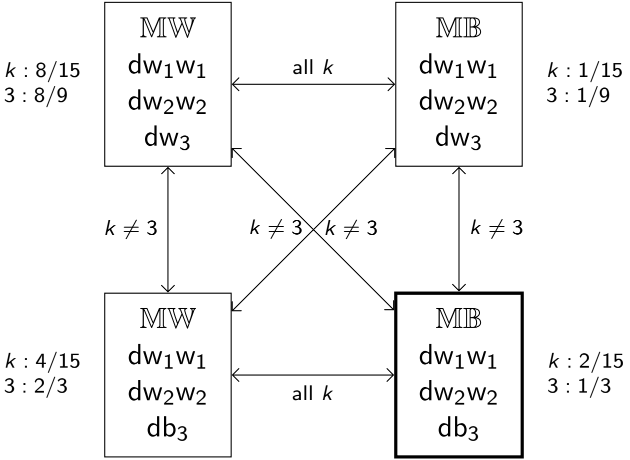


Agent 2 considers the majority white urn to be more likely, and will write white.

The result of agent 2's guess is bisimilar (via generated submodel) to the following:

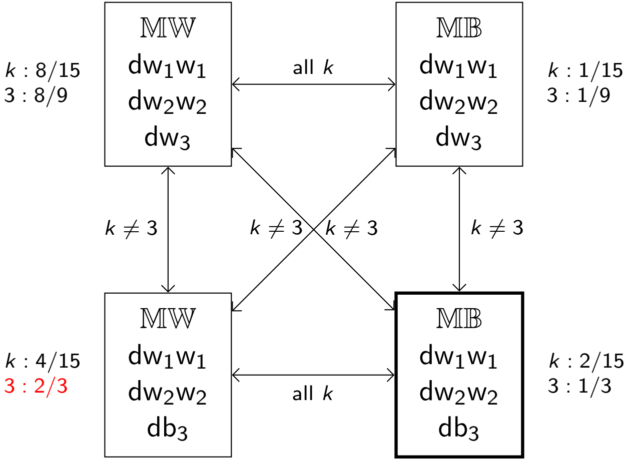


Result of agent 3 drawing black



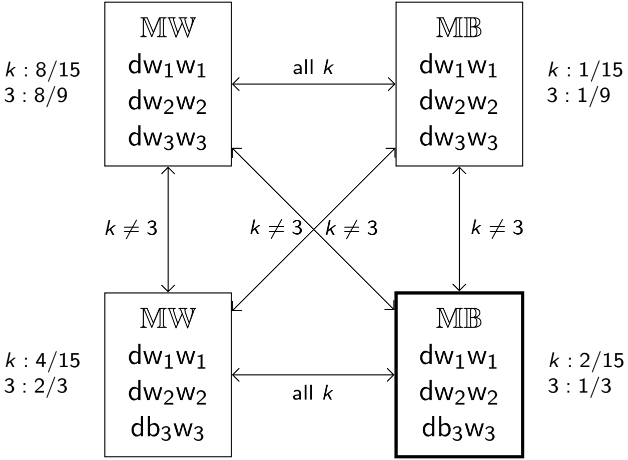
Agent 3 considers the majority white urn to be more likely, and will write white.

Result of agent 3 drawing black



Agent 3 considers the majority white urn to be more likely, and will write white.

Result of agent 3 writing

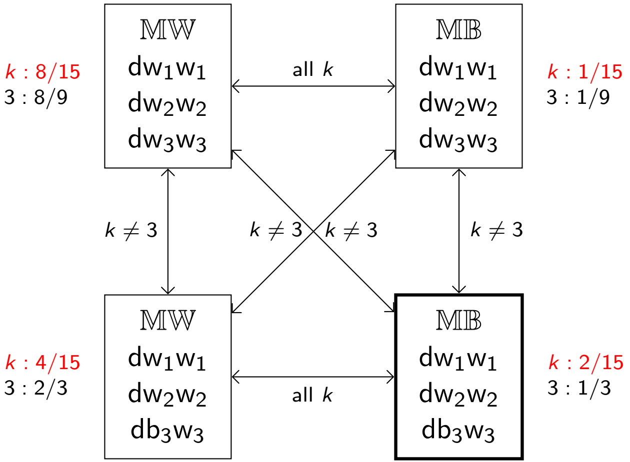


$\models P_k(\mathrm{MW}) = 12/15 \wedge P_k(\mathrm{MB}) = 3/15.$

$(\models P_k(\mathrm{MW}) = 4/5 \wedge P_k(\mathrm{MB}) = 1/5.)$

This is similar to the situation after agent 2 wrote.

Result of agent 3 writing



$\models P_k(\text{MW}) = 12/15 \wedge P_k(\text{MB}) = 3/15.$

$(\models P_k(\text{MW}) = 4/5 \wedge P_k(\text{MB}) = 1/5.)$

This is similar to the situation after agent 2 wrote.

Reduction axioms

- $[e]p \leftrightarrow (\mathrm{pre}(e) \rightarrow \mathrm{sub}(e)(p))$
- $[e]\neg\phi \leftrightarrow (\mathrm{pre}(e) \rightarrow \neg[e]\phi)$
- $[e](\phi \wedge \psi) \leftrightarrow ([e]\phi \wedge [e]\psi)$
- $[e][\pi]\phi \leftrightarrow \bigwedge_{f \in E}[T_{ef}\pi][f]\phi$

  T_{ef} is a program transformer whose definition depends on test programs $[?\phi]$.
- $[e]\left(\sum_{k=1}^{n} \alpha_k \cdot P_a(\psi_k) \geq \beta\right) \leftrightarrow (\mathrm{pre}(e) \rightarrow C \geq D)$, where

$$C = \sum_{\phi\in\Phi} \sum_{f\sim_a e} \sum_{k=1}^{n} \alpha_k \cdot \mathrm{pre}_a(f \mid \phi) \cdot P_a(\phi \wedge [f]\psi_k),$$
$$D = \sum_{\phi\in\Phi} \sum_{f\sim_a e} \beta \cdot \mathrm{pre}_a(f \mid \phi) \cdot P_a(\phi).$$

Completeness

Theorem (Weak completeness)

For all $\varphi \in \mathcal{L}_{PLCC}$,

$$\vdash \varphi \textit{ if and only if } \models \varphi$$

- Proof by reduction Proof reduces each PLCC formula to a provably equivalent PE-PDL formula.
  Completeness then follows from completeness of PE-PDL
- Decidability follows from computability of the reduction and finite model construction in the completeness proof of PE-PDL
- Although PLCC and PE-PDL are equi-expressive, PLCC formulas are often more succinct than PE-PDL formulas.

Conclusion

- We provided a dynamic probabilistic (epistemic) logic that can express common knowledge:
- We expect that this language will be used for other examples:
  - Variations of the Urn Example
  - Other examples of informational cascade
  - Other multi-agent probabilistic examples
- Language can be adapted further to incorporate preferences or payoffs

A Key Observation

Higher order reasoning (even common knowledge) cannot necessarily prevent false cascades.

THANK YOU!